THESIS

FOR

THE DEGREE OF D.SC.

'ON THE RELATION OF MEAN STELLAR PARALLAX TO PROPER MOTION AND APPARENT MAGNITUDE'

AND OTHER PAPERS

Thesis presented for the degree of D.Sc.

1925

BY

GORAKH PRASAD

As required by Regulation XII. of the Regulations regarding Graduation in Pure Science (Edinburgh University Court Ordinance No. 35) I hereby declare that the work has been done and the thesis composed by myself.

1925 January 14.

Contents

On the Relation of Mean Stellar Parallax to Proper Motion and Apparent Magnitude.

Advance proof from the Monthly Notices
of R.A.S., December, 1924.

On the Relation of Mean Parallax to Proper Motion and Apparent Magnitude. By Gorakh Prasad, M.Sc.

(*Communicated by the Astronomer Royal for Scotland.*)

The relation between mean parallax, proper motion, and apparent magnitude has been studied by several investigators. Kapteyn and van Rhijn * showed that

$$\log \bar{\pi} = -0{\cdot}690 - 0{\cdot}0713m + 0{\cdot}645 \log \mu \quad . \quad . \quad (1)$$

Luyten † has pointed out that the numerical constants entering in this formula are such that the latter may be replaced, practically without any error, by a simpler relation of the form

$$\bar{M} = a + bH \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where a and b are constants and

$$H = m + 5 + 5 \log \mu \quad . \quad . \quad . \quad . \quad (3)$$

The discussion by Luyten ‡ of such stars from the list of spectroscopic parallaxes for 1646 stars given by Adams, Joy, Strömberg and Burwell § as had also a known trigonometrical parallax shows that the relation (2) is not at variance with facts when a and b are made dependent on spectral class. Luyten does not consider values of H much greater than 12. Using 325 stars having $\mu \geqq 0''{\cdot}5$ and 181 stars having $0''{\cdot}3 \leqq \mu < 0''{\cdot}5$, and treating them together, irrespective of their spectral types, Seares ‖ comes to the conclusion that the relation of $\bar{M}$ to H is not linear, although a linear formula can be used on either side of the value $H = 12{\cdot}5$.

Thus Luyten and Seares both agree that up to $H = 12{\cdot}5$ the relation between H and $\bar{M}$ is linear. The present investigation was undertaken to obtain improved values of the constants in relation (2) by using the trigonometrical parallaxes of a much larger number of stars and treating them by an improved method. This has been done. The probable errors of the values of the constants as obtained in this paper are much smaller than the corresponding probable errors of the values found by Luyten. The investigation shows, moreover, that for the F and G stars the relation between H and $\bar{M}$ is not linear.

1. *Formula for calculating the Mean Absolute Magnitude.*

The method employed by Luyten consists in grouping the stars with respect to H. The absolute magnitude of every star is calculated from its observed parallax, and then the median values of the absolute magnitudes and H are derived for every group and used to solve the equations (2). Luyten points out that the use of the median value has the advantage of allowing stars with negative parallaxes to be used, for the absolute magnitude of any such star may be put down as $-\infty$. Seares simply takes the arithmetic means of H and the absolute magnitudes, no negative parallaxes being used. Obviously the plan of taking the mean of the calculated absolute magnitudes cannot be used with the material under discussion, for so many of the trigonometrical parallaxes come out

References on next page.

as negative. In order to make use of these negative parallaxes more effectively, we shall take the arithmetic mean of the measured parallaxes π_0, the groups being formed with respect to H as usual, and calculate $\bar{M}$ by the following formula due to Strömberg ¶:

$$\bar{M} = 5 - 5 \log \overline{10^{-0 \cdot 2m}} + 5 \log \bar{\pi}_0 \quad . \quad . \quad . \quad (4)$$

The advantage of using this formula instead of the median is considered further on.

2. *Material Used.*

The present discussion is based on the trigonometrical parallaxes determined by photography at the Allegheny, McCormick, Yerkes, Greenwich, Sproul, Mount Wilson, and Dearborn Observatories. Out of the parallaxes published up to the present time, about 2200 in number, some relate to stars of which the spectral type is not known. Others could not be included in the present investigation, because they relate to stars whose total proper motion is not known. The available number of stars and parallax determinations in the case of each spectral type is shown in columns 2 and 3 of Table I. In cases where there are two or more determinations for the same star, the simple arithmetic mean of the various values has been taken as the parallax of the star. No regard has been paid to the probable errors of the parallaxes in forming the mean, because even the various determinations with fairly small probable errors are often very discordant. No parallaxes have been discarded on account of their large negative values or on account of their large probable errors. A few of the parallaxes have probable errors of $0''\cdot020$ or more. These might have been rejected, but in view of the fact that in general the true probable errors are greater than the published values and the latter are themselves affected with accidental errors, it has not been done. In the case of binaries the components have been treated as distinct stars. This has been done because in some of the binaries the companion belongs to a spectral type different from that of the primary; but only the common proper motion has been used in calculating H. The magnitudes used are all visual and on the Harvard scale. In the case of double and binary stars, if the separate magnitudes are not given in the Henry Draper Catalogue, these have been calculated from the combined magnitude as given in this catalogue and the difference of magnitude as given by Burnham. The spectral types used are those given in the Henry Draper Catalogue. In the case of double and binary stars, where the separate spectra were not given in H.D., the spectrum of the companion was assumed to be the same as that of the brighter star if the difference in magnitude was not more than 1, the combined magnitude was not fainter than 7·0, and the combined spectrum was not marked as peculiar. This has not been done very often; for example, in the case of the G stars this had

* *Mount Wilson Contr.*, No. 188; *Astroph. Journ.*, **52**, 23, 1920.
† *Lick Obs. Bull.*, **11**, 39 (No. 345), 1923.
‡ *Lick Obs. Bull.*, **10**, 135 (No. 336), 1922.
§ *Mount Wilson Contr.*, No. 199; *Astroph. Journ.*, **53**, 13, 1921.
‖ *Mount Wilson Contr.*, No. 273; *Astroph. Journ.*, **59**, 310, 1924.
¶ See *Astroph. Journ.*, **57**, 296.

to be done in seven cases only. For some stars the Mount Wilson estimated spectrum was used, the H.D. spectrum not being available. Such cases are few in number. Thus in the case of type G only three stars (companions to bright stars) were taken to be of type G on the authority of the Mount Wilson spectrum. In the case of the variable stars the mean magnitude has been used to calculate H and M, if the variation in apparent magnitude is less than one magnitude; otherwise the star has not been included in the discussion. The proper motions are generally from Boss or *Cincinnati Publication*, No. 18; but for the Greenwich stars, if the proper motion could not be derived from these sources, the total proper motion was calculated from the components (those derived from catalogue positions) published with the parallaxes. Only in the case of A stars eight of the proper motions had to be taken from various sources as given in Shorr's *Eigenbewegungs Lexikon*. Stars with $\mu < 0''\cdot005$ were excluded.

Van Maanen and Miss Wolfe * have studied the systematic differences in the parallaxes determined at different observatories. Some of the corrections deduced by them are pretty large. For the Greenwich parallaxes the correction amounts to $0''\cdot012$, the probable error of this value of the correction being only $0''\cdot001$. Strömberg † also finds for the observatories he considers similar corrections. Still, these corrections were not applied to the parallaxes used in this paper, because of the comparative scantiness of the material used in the investigation of Van Maanen and Miss Wolfe and the greater uncertainty of the corrections derived by Strömberg. Moreover, the only effect which these systematic errors can have is to cause a slightly greater dispersion in the values of M. To convert relative parallaxes into absolute ones, a constant reduction of $0''\cdot002$ has been applied to the parallaxes determined at the Mount Wilson Observatory, and $+0''\cdot005$ to the parallaxes determined at the other observatories.‡

3. *Relation between* H *and* $\bar{M}$.

The stars of each spectral type were divided into a number of groups, each group except either the first (that with the smallest value of H) or the last (that with the largest value of H) containing the same number of stars. If the total number of stars exceeded an exact multiple of the number of groups by less than five, these extra number of stars was included in the first group; otherwise the last group contained a smaller number of stars than the other groups. The mean values of H and of M as calculated from formula (4) for these groups are shown in Tables II. to X., and plotted in figs. 1 to 6. In the case of the F and G stars for the range of values of H from 1 to 12 and 1 to 14 respectively, it was assumed that the relation between H and $\bar{M}$ is

$$\bar{M} = a + bH + cH^2 \quad . \quad . \quad . \quad . \quad (5)$$

In all other cases it was assumed that this relation is

$$\bar{M} = a + bH \quad . \quad . \quad . \quad . \quad . \quad (6)$$

References on next page

The values of a, b, c were found by the Method of Least Squares, and they are shown, together with their probable errors, in Table I. The corresponding curves or straight lines are shown by continuous lines in the figures. The columns headed M_c or M'_c in Tables II. to X. give the value of the mean absolute magnitude as calculated from equation (5) or (6) with the appropriate values of a, b, c. The residuals are also shown in the tables.

TABLE I.

The Constants in the Relation between H *and* $\overline{M}$.

Spectral Class.	Number of Stars.	Number of Parallaxes.	Number of Groups.	Range of the Values of H.	a.	b.	c.
B8, B9, A0, A2, A3, A5	190	257	5	0–7	$-0.43\pm.19$	$.407\pm.14$	
F0, F2, F5, F8	313	453	10	1–12	$-0.92\pm.14$	$.762\pm.044$	$-.0223\pm.0031$
F0, F2, F5, F8	248	361	8	6–12	$+0.90\pm.19$	$.357\pm.022$	
G0, G5	382	517	10	1–14	$-2.11\pm.18$	$.919\pm.053$	$-.0223\pm.0037$
G0, G5	304	412	8	7–14	$+0.11\pm.38$	$.469\pm.036$	
K0, K2 Giants	155	202	5	0–7	$-0.35\pm.09$	$.307\pm.022$	
K0, K2, K5 Giants	185	237	5	0–7	$-0.59\pm.14$	$.364\pm.032$	
K0, K2 Dwarfs	153	219	5	8–14	$-3.57\pm.51$	$.866\pm.045$	
K0, K2 Dwarfs	184	258	6	6–14	$-4.04\pm.31$	$.905\pm.091$	
K5, Ma, Mb, Mc Giants	80	103	6	−1–7	$-1.84\pm.23$	$.647\pm.056$	
Ma, Mb, Mc Giants	50	68	4	0–7	$-1.41\pm.27$	$.435\pm.062$	
K5, Ma, Mb, Mc Dwarfs	56	94	4	11–16	$+0.57\pm1.71$	$.601\pm.39$	

TABLE II.

A Stars.

Mean Values of H and M.

Number.	$\overline{H}$.	$\overline{M}$.	$\overline{M}_c$.	$\overline{M}_c-\overline{M}$.
1	0·88	0·22	−0·07	−·29
2	2·94	0·43	0·77	+·34
3	3·97	1·21	1·19	−·02
4	4·94	1·52	1·58	+·06
5	6·77	2·54	2·33	−·21

* *Mount Wilson Contr.*, No. 189.
† *Ibid.*, No. 220.
‡ Cf. *Mount Wilson Contr.*, No. 199, p. 7.

TABLE III.

F Stars.

Mean Values of H and M.

Number.	$\overline{H}$.	$\overline{M}$.	$\overline{M}_c$ from Curve.	$\overline{M}'_c$ from Straight Line.	$\overline{M}_c - \overline{M}$.	$\overline{M}'_c - \overline{M}$.
1	1·59	0·22	0·23		+·01	
2	4·97	2·29	2·32		+·03	
3	6·06	2·89	2·89	3·06	·00	+·17
4	6·79	3·50	3·23	3·32	−·27	−·18
5	7·34	3·38	3·47	3·52	+·09	+·14
6	7·84	3·51	3·69	3·70	+·18	+·19
7	8·32	3·97	3·88	3·87	−·09	−·10
8	9·00	4·17	4·14	4·11	−·03	−·06
9	9·87	4·28	4·44	4·42	+·16	+·14
10	11·94	5·07	5·02	5·16	−·05	+·09

TABLE IV.

G Stars.

Mean Values of H and M.

Number.	$\overline{H}$.	$\overline{M}$.	$\overline{M}_c$ from Curve.	$\overline{M}'_c$ from Straight Line.	$\overline{M}_c - \overline{M}$.	$\overline{M}'_c - \overline{M}$.
1	1·15	−1·04	−1·09		−·05	
2	4·99	1·79	1·92		+·13	
3	7·65	3·66	3·62	3·70	−·04	+·04
4	8·79	4·03	4·25	4·23	+·22	+·20
5	9·37	4·94	4·54	4·50	−·40	−·44
6	9·91	4·78	4·81	4·76	+·03	−·02
7	10·41	5·15	5·04	4·99	−·11	−·16
8	10·95	5·29	5·28	5·24	−·01	−·05
9	11·58	5·27	5·54	5·54	+·27	+·27
10	13·14	6·17	6·12	6·27	−·05	+·10

TABLE V.

*K*o, *K*2 *Giants.*

Mean Values of H and M.

Number.	$\overline{H}$.	$\overline{M}$.	$\overline{M}_c$.	$\overline{M}_c - \overline{M}$.
1	0·56	−0·10	−0·18	−·08
2	3·02	0·48	0·58	+·10
3	4·23	0·79	0·95	+·16
4	5·15	1·38	1·23	−·15
5	6·43	1·64	1·62	−·02

TABLE VI.

Ko, K2, K5 Giants.

Mean Values of H and M.

Number.	$\overline{H}$.	$\overline{M}$.	$\overline{M}_c$.	$-\overline{M}_c+\overline{M}$.
1	0·49	−0·32	−0·41	+·09
2	2·77	0·43	0·42	+·01
3	4·09	0·58	0·90	−·32
4	5·05	1·35	1·25	+·10
5	6·34	1·83	1·72	+·11

TABLE VII.

Ko, K2 Dwarfs.

Mean Values of H and M.

Number.	$\overline{H}$.	$\overline{M}$.	$\overline{M}_c$ from First Straight Line.	$\overline{M}'_c$ from Second Straight Line.	$\overline{M}_c-\overline{M}$.	$\overline{M}'_c-\overline{M}$.
1	6·43	1·64		1·78		+·14
2	8·25	3·40	3·57	3·43	+·17	+·03
3	10·12	5·58	5·19	5·12	−·39	−·46
4	11·23	6·10	6·15	6·12	+·05	+·02
5	12·05	6·66	6·87	6·87	+·21	+·21
6	13·75	8·36	8·34	8·40	−·02	+·04

TABLE VIII.

K5, Ma, Mb, Mc Giants.

Mean Values of H and M.

Number.	$\overline{H}$.	$\overline{M}$.	$\overline{M}_c$.	$\overline{M}_c-\overline{M}$.
1	−0·20	−2·15	−1·97	+·18
2	2·02	+0·03	−0·53	−·56
3	3·04	−0·37	0·13	+·50
4	4·20	+0·84	0·88	+·04
5	5·13	+1·86	1·48	−·38
6	6·40	+2·08	2·30	+·22

TABLE IX.

Ma, Mb, Mc Giants.

Mean Values of H and M.

Number	$\bar{H}$.	$\bar{M}$.	$\bar{M}_c$.	$\bar{M}_c - \bar{M}$.
1	0·24	−1·32	−1·31	+·01
2	3·24	−0·25	0·00	+·25
3	4·65	1·09	0·61	−·48
4	6·35	1·11	1·35	+·24

TABLE X.

K5, Ma, Mb, Mc Dwarfs.

Mean Values of H and M.

Number.	$\bar{H}$.	$\bar{M}$.	$\bar{M}_c$.	$\bar{M}_c - \bar{M}$.
1	11·64	7·80	7·57	−·23
2	13·32	8·33	8·58	+·25
3	14·54	8·97	9·31	+·34
4	15·96	10·45	10·16	−·29

The various special points relating to the different spectral types will now be briefly taken up.

A Stars.—In this division all the A stars and also B8 and B9 stars have been included. Three stars with exceptionally large values of H, viz. 9·3, 12·8, and 17·8, have been excluded. As the parallaxes and proper motions are relatively small for the A stars, the relation between H and $\bar{M}$ for them is more uncertain than the number of stars would lead us to expect. At first these stars were divided into ten groups, but the points thus obtained were more or less irregularly scattered. The points obtained by dividing the stars into five groups are more satisfactory, as can be seen from fig. 1. These stars were not investigated by Luyten.

~~Fig. 1.—A stars.~~

F Stars.—This division includes all the F subdivisions, viz. F0, F2, F5, and F8. One star with an exceptionally large value of H, viz. H = 19·7, was excluded. The relation between H and $\bar{M}$ is seen from fig. 2 to be non-linear. This non-linearity is indicated by Luyten's work

~~Fig. 2.—F stars.~~

also, for the points for the F stars in fig. 1 of his paper * can be better represented by a curve than a straight line. The equation of the best straight line passing through the eight points with $\bar{H} > 6$ is also given in Table I. It is discussed below in connection with a similar straight line for the G stars.

G Stars.—This division includes both the subdivisions G0 and G5. Here also the relation between H and $\bar{M}$ is seen to be non-linear. The curve of the second degree shown in fig. 3 is seen to fit the points satis-

References on p. 10.

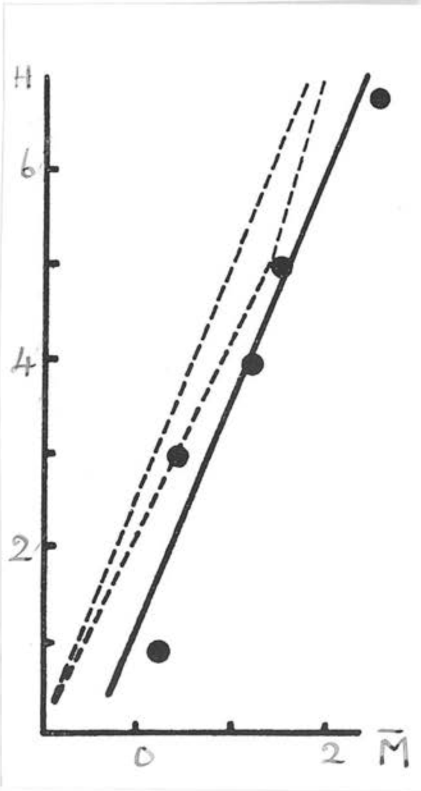

Fig. 1. — A stars.

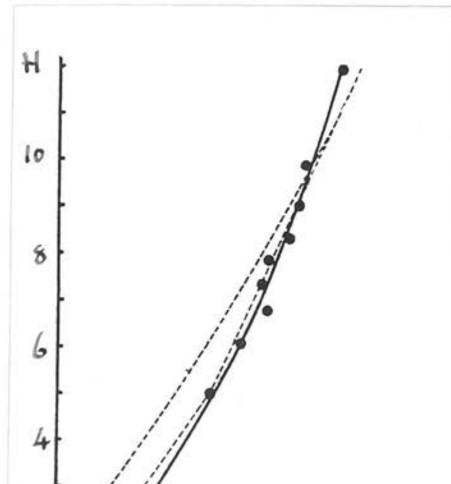

Fig. 2. — F stars.

factorily. A glance at fig. 2 of Luyten's paper will show that his points do not fit a straight line, or in fact any simple curve, so closely. This must be due to the smaller number (253) of the stars considered, to the inclusion of a number of the earlier and less reliable parallaxes, and to the use of the median instead of the mean value.

In the paper by Seares, no mean values of H less than + 5·9 have been used.† The relation between H and $\bar{M}$ for values of H from this up to 12·5 is found by him to be linear, as stated before. The first two groups of Table IV. were therefore omitted, values of H from 7·6 to 13·1 only being thus taken into consideration, and the equation of the best straight line through the remaining eight points was found. This is given in Table I. The regularity in the sign of the residuals (see Table IV.) shows that curvature is perceptible even in this short range. Moreover, the sum, 0·35, of the squares of the residuals is greater than the sum, 0·30, of the squares of the corresponding residuals for the curve. It appears, therefore, that the relation between H and $\bar{M}$ for the G stars should preferably not be taken as linear. A similar remark applies to the F stars. Here the residuals for the straight line passing through the eight points with the largest values of H do not show the same regularity in the signs of the residuals, but still the sum, 0·16, of the squares of these residuals is slightly greater than the corresponding sum, 0·15, for the curve.

K Stars.—This division includes only K0 and K2 stars. The points obtained are shown in fig. 4. It is evident that two straight lines will

~~Fig. 4.—K0 and K2 stars.~~

give a better representation than any curve of the second degree. It seems that the point H = 6·4, M = 1·6 lies on both the straight lines and could ~~have been~~ included very well in the computations of both. Unlike the case of the M stars considered below, there is no sharp gap in the values of H to indicate a point of division. Eddington and Miss Douglas ‡ have taken M = 3·5 to be the dividing line between the K giants and dwarfs. From fig. 4 of this paper it is evident that the point H = 8·3, M = 3·4 should be included among the points for dwarfs. The point in question, with a value of M much less than 3·5, was therefore included in the discussion of the giants only. But a second solution for the dwarfs has also been made with this point included. The equation of the straight line thus obtained, which is almost coincident with the former one, is given in Table I. and the residuals in Table VII. Luyten also has considered the giants and dwarfs separately, but his straight line for the dwarfs is very different from the line obtained here, as is shown in fig. 4.

* *Lick Obs. Bull.*, **10**, 137.

† It should be noted that the notation adopted in the present paper is that of Luyten's second paper (*Lick Obs. Bull.*, **11**, 39, No. 345). In the paper by Seares H stands for $m + \log_{10} \mu$.

‡ *Monthly Notices*, **83**, 115.

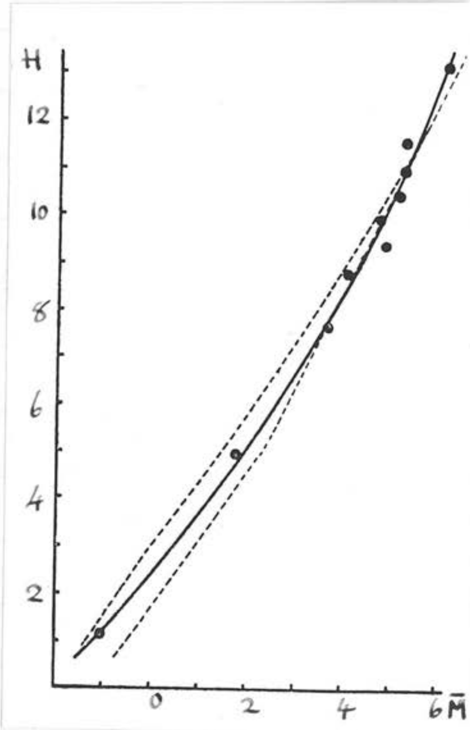

Fig. 3. — G stars.

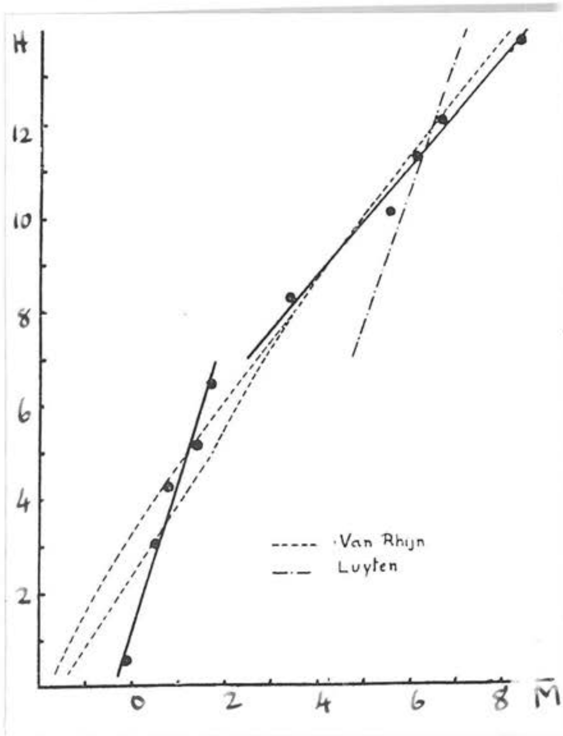

Fig. 4. — K0 and K2 stars.

M Stars.—This division includes K5 and M stars. Barnard's Proper Motion star was excluded on account of the abnormally large value of its H. There is a sharp division between the giants and dwarfs marked by the absence of stars with a value of H between 8·1 and 10·2. The number of stars is not large enough to give a very satisfactory determination, and the probable errors of the constants are larger. In the case of the giants the straight line obtained for these stars differs considerably from the straight line obtained, as explained in Art. 8, from Van Rhijn's mean parallaxes for M stars. To see how far this difference was due to the inclusion of K5 stars, a second solution was made from which all the K5 stars were excluded. The straight line thus obtained is much nearer Van Rhijn's line, as is shown by fig. 5.

~~Fig. 5.—M Giants. The dots represent groups of K5 and M stars, the circles those of M stars alone.~~

In the case of the dwarfs the number of stars is smaller still, and consequently the relation between H and M is somewhat uncertain. It is interesting to note that Barnard's Proper Motion star lies fairly near the straight line determined from the rest of the stars (fig. 6). The straight line for the giants is nearly parallel to that of the dwarfs, but the one is not a continuation of the other, as can be seen from fig. 7.

~~Fig. 6.—M Dwarfs. The circle represents Barnard's Proper Motion star.~~

~~Fig. 7.—M stars.~~

4. *Effect of Accidental Errors.*

It has been pointed out by Seares * that accidental errors in any quantity produce a systematic error in the mean of a number of values of this quantity if the frequency distribution of these values is not uniform. In view of this it seems necessary to examine the magnitude of the systematic errors in the mean values of H in the various groups. Fortunately, the accidental errors of the proper motions are much smaller than the accidental errors of the parallaxes, and therefore the errors in H are relatively unimportant. Since an error $\delta\mu$ in μ produces an error δH in H where approximately

$$\delta H = 2{\cdot}2\,\delta\mu/\mu,$$

the largest errors for each spectral type are likely to occur in group 1, which contains the smallest proper motions. There is another factor tending towards the same effect, namely, the great non-uniformity of the frequency distribution of the values of H within and in the neighbourhood of this group. Hence it will suffice to make sure that the systematic errors in such a group are negligible.

Since H is the same function of μ as M is of π, we find, almost exactly as in the case of absolute magnitudes discussed by me in a previous paper, † that the error in $\bar{H}$ is

References on p. 15.

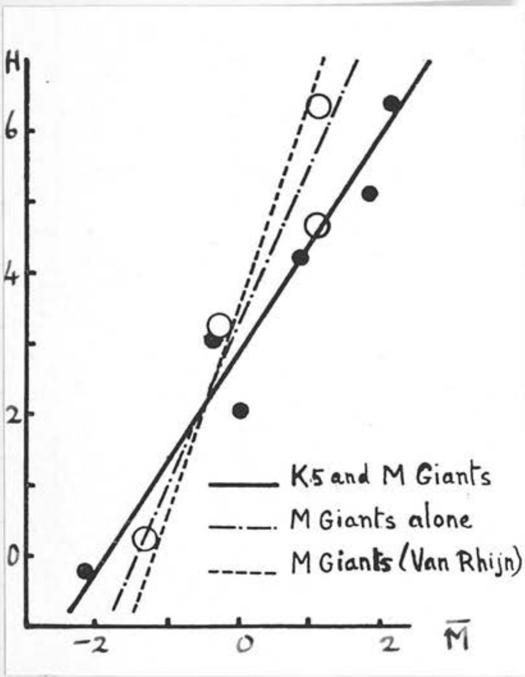

Fig. 5. — M Giants. The dots represent groups of K5 & M stars, the circles those of M stars alone.

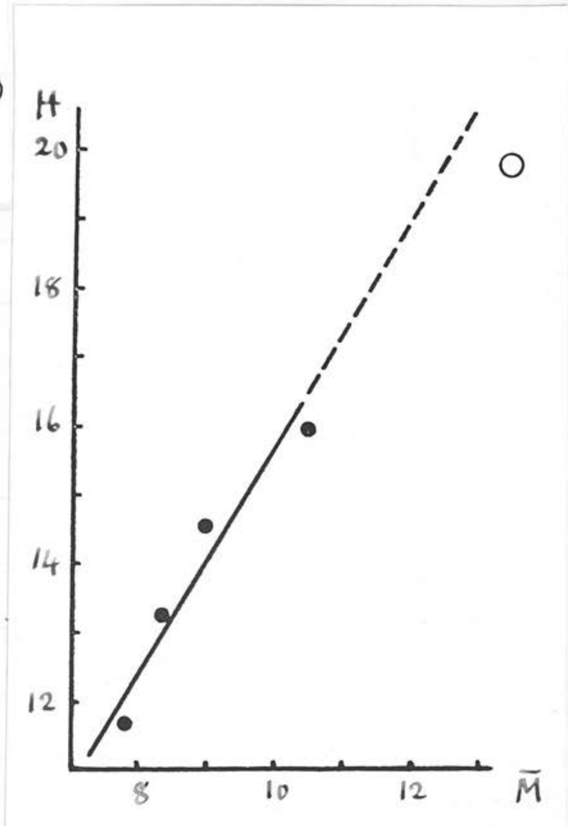

Fig. 6. — M Dwarfs. The circle represents Barnard's Proper Motion star.

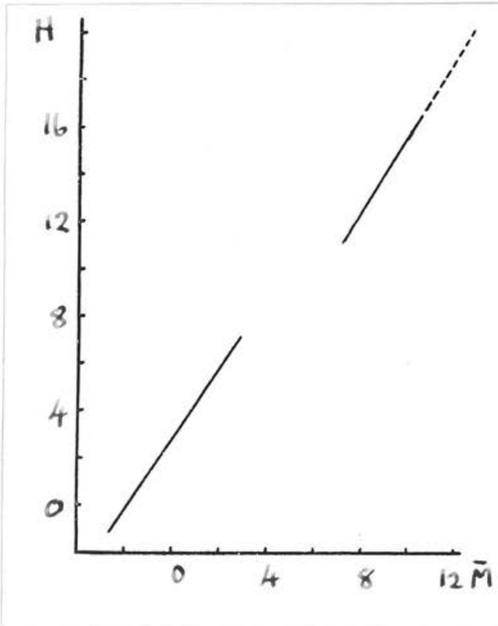

Fig. 7. — M stars.

$$\frac{5Nh_1k^{-\frac{1}{2}}d\mathrm{H}\int_g^\infty\int_0^\infty u(\mu)e^{-h_1^2(\mu-\mu_1)^2}f(\mu_1,\mathrm{H})(\log\mu-\log\mu_1)d\mu d\mu_1}{Nh_1k^{-\frac{1}{2}}d\mathrm{H}\int_g^\infty\int_0^\infty u(\mu)e^{-h_1^2(\mu-\mu_1)^2}f(\mu_1,\mathrm{H})d\mu d\mu_1},$$

where N is the total number of stars considered,

h_1 is the modulus of precision of the proper motions,

k is the ratio of the circumference of a circle to its diameter,

dH is the difference between the largest and the smallest value of H in the group under consideration,

$u(\mu)$ represents the true frequency distribution of the proper motions, so that $Nu(\mu)d\mu$ is the number of stars having a true proper motion between μ and $\mu+d\mu$,

$f(\mu_1, \mathrm{H})$ is the number of stars which have their measured proper motion between μ_1 and $\mu_1+d\mu_1$ and their H (calculated from the measured proper motions) between H and H + dH divided by the total number of stars between the same limits of proper motion irrespective of their value of H,

and g is the smallest value of the measured proper motion considered.

We consider now an actual example, say group 1 of the G stars. Omitting the 5 stars with a measured proper motion of less than 0″·010 per year, assuming that no star with a true proper motion of less than 0″·010 per year occurs in this group (some such supposition is necessary, for if a single star with a true proper motion equal to zero were to be included in this group the error in $\bar{\mathrm{H}}$ would rise to infinity), and taking the probable error of the proper motions of this group to be 0″·006 per year, we find on computation that the error in $\bar{\mathrm{H}}$ is less than 0·01 in numerical value. We may conclude therefore that no appreciable systematic errors exist in the mean values of H.

In order to see if the very small proper motions had any harmful effect on the value of $\bar{\mathrm{H}}$ for this particular group, the values of $\bar{\mathrm{H}}$ and $\bar{\mathrm{M}}$ were computed after excluding (i) all the 5 stars with $\mu<0''\cdot010$, and (ii) all the 15 stars with $\mu<0''\cdot020$. The points corresponding to these groupings ($\bar{\mathrm{H}}=1\cdot37$, $\bar{\mathrm{M}}=-1\cdot13$) and ($\bar{\mathrm{H}}=2\cdot02$, $\bar{\mathrm{M}}=-0\cdot72$) lie fairly near the curve, the residuals being + ·24 and + ·38.

Though not connected with the present topic, it may be mentioned here that the last group was re-calculated by omitting the 6 stars with $\mu>2''\cdot00$, to find if the very large proper motions had any abnormal effect. The result is $\bar{\mathrm{H}}=12\cdot83$, $\bar{\mathrm{M}}=6\cdot06$. This gives a residual of − ·04 which is satisfactorily small.

Seares ‡ states that "there is always danger of trouble in attempting to determine the relation of absolute magnitude to any other characteristic, x, even when x itself is free from error." While this is true when the groups are formed with respect to the absolute magnitudes, it is obvious that it is not true if the groups are formed with respect to x. Since the groups above have been formed according to $\bar{\mathrm{H}}$, the accidental errors of the parallaxes will produce no systematic errors in

* *Monthly Notices*, 84, 15.

† *Ibid.*, 84, 493. (p. 41 of this thesis).

‡ *Monthly Notices*, 84, 19.

the mean absolute magnitudes. The apparent magnitudes, m, are used in the calculation of both H and M, and the accidental errors in m will produce systematic errors in both $\bar{\mathrm{H}}$ and $\bar{\mathrm{M}}$, but these must be very small.

5. *Systematic Errors.*

Since in formula (4) only $\bar{\pi}_0$ is used, the effect of the accidental errors in the parallaxes is much reduced. But the formula is not strictly true. If for each individual star the relation

$$\mathrm{M} = a + b\mathrm{H} + c\mathrm{H}^2 \quad . \quad . \quad . \quad . \quad (7)$$

were valid (and if the mean of the observed parallaxes π were equal to the mean of the true parallaxes π, which may be assumed to be true for the purposes in hand), the value of $\bar{\mathrm{M}}$ given by (4) would be quite correct. This is easily seen by expressing π in terms of H and m from (7) and taking the logarithm of the mean. But equation (7) is not satisfied individually, and we are to calculate a, b, and c on the assumption that the relation $\bar{\mathrm{M}} = a + b\mathrm{H} + c\mathrm{H}^2$ holds for a group consisting of a large number of stars. We can estimate the error introduced by the use of (4) as follows:—

Consider any one group consisting of a large number, say n, of stars, and let $a + b\mathrm{H} + c\mathrm{H}^2$ have the value M for this group. Let the true values of the absolute magnitudes of the stars of the group be denoted by M′. Put

$$\mathrm{M} = \mathrm{M}' + x \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We suppose now that x has a normal frequency distribution, so that the number of stars for which x has a value between x and $x + dx$ is

$$nhk^{-\frac{1}{2}}e^{-h^2x^2}dx \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where k is the ratio of the circumference of a circle to its diameter as before. Then $\bar{\mathrm{M}}' = \mathrm{M}$, and the correction to be added to the mean absolute magnitude calculated from (4) is

$$\mathrm{M} - \{5 - 5 \log \overline{10^{-0{\cdot}2m}} + 5 \log \bar{\pi}\} \quad . \quad . \quad . \quad (10)$$

Substituting $10^{0{\cdot}2(\mathrm{M}-x-5-m)}$ for π, this reduces to

$$5 \log \overline{10^{-0{\cdot}2m}} - 5 \log \overline{10^{-0{\cdot}2x} \,.\, 10^{-0{\cdot}2m}}.$$

Now x and m can be supposed to be independent. This gives

$$\overline{10^{-0{\cdot}2x} \,.\, 10^{-0{\cdot}2m}} = \overline{10^{-0{\cdot}2x}} \,.\, \overline{10^{-0{\cdot}2m}},$$

and the correction (10) thus becomes

$$-5 \log \overline{10^{-0{\cdot}2x}}.$$

From (9) we have

$$\overline{10^{-0{\cdot}2x}} = \frac{h}{k^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{-h^2x^2}\, 10^{-0{\cdot}2x}dx$$
$$= e^{0{\cdot}01/h^2\mathrm{Mod}^2}.$$

Therefore the correction to the mean absolute magnitude calculated from (4) is, finally,

$$-\frac{0{\cdot}05}{\text{Mod}}\cdot\frac{1}{h^2} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This expression holds also when the parallaxes are considered to be affected by accidental errors and also if $c=0$ in (7).

Thus the error committed by the use of (4) is small if the dispersion is small, but a correction can be applied if necessary. The value of h is not easy to evaluate with certainty, and this is a disadvantage, but the use of the median of the individually calculated absolute magnitudes is also affected with a systematic error, as can be seen from what follows.

6. *Systematic Error of the Median Value.*

Consider the same group of stars as above, and let the absolute magnitudes calculated from the measured parallaxes π_0 be denoted by M_0. Put

$$M=M_0+y \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The differences y between M and M_0 consist of two parts, one part, x, due to the dispersion of the true absolute magnitudes M' about M, and another part, say ξ, due to the errors in M_0 caused by the accidental errors in the parallaxes. Let

$$\pi=\pi_0+z \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where the accidental errors z are supposed to be distributed normally, the modulus of precision of π_0 being denoted by h'. Since

$$M'=m+5+5\log\pi \quad . \quad . \quad . \quad . \quad (14)$$

and

$$M_0=m+5+5\log\pi_0, \quad . \quad . \quad . \quad . \quad (15)$$

we have

$$\xi=M'-M_0=5\log(\pi/\pi_0)=-5\log(1-z/\pi) \quad . \quad . \quad (16)$$

The frequency distribution of ξ in a sub-group consisting of all the stars having their true parallaxes between π and $\pi+d\pi$ is therefore represented by

$$0{\cdot}2h'\pi k^{-\frac{1}{2}}(\text{Mod})^{-1}e^{-h'^2\pi^2(1-10^{-0{\cdot}2\xi})^2}10^{-0{\cdot}2\xi}d\xi \quad . \quad . \quad (17)$$

together with $n_1 p$ stars at $+\infty$, where

$$p=\frac{h'}{k^{\frac{1}{2}}}\int_{\pi}^{\infty}e^{-h'^2x^2}dx \quad . \quad . \quad . \quad . \quad (17a)$$

and n_1 is the number of stars in the sub-group. Thus

$$y=x+\xi \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where x is distributed according to (9) and ξ is distributed according to (17). The frequency distribution of y is therefore given by $\Phi(y)$, where *

$$\Phi(y)=\frac{1}{2k}\int_{-\infty}^{\infty}\Omega_1(\Theta)\Omega_2(\Theta)e^{i\Theta y}d\Theta \quad . \quad . \quad . \quad (19)$$

where

$$\Omega_1(\Theta)=\frac{h}{k^{\frac{1}{2}}}\int_{-\infty}^{\infty}e^{-i\Theta x}e^{-h^2x^2}dx=e^{-\Theta^2/4h^2}$$

and

$$\Omega_2(\Theta)=\frac{0{\cdot}2\pi h'}{k^{\frac{1}{2}}\,\mathrm{Mod}}\int_{-\infty}^{\infty}e^{-i\Theta\xi}e^{-h'^2\pi^2(1-10^{-0\cdot2\xi})^2}10^{-0\cdot2\xi}d\xi.$$

This is the distribution of ξ within the sub-group with a constant π. If the frequency distribution of π in the group is given by $f(\pi)$, where $f(\pi)$ can be calculated from the observed frequency distribution, $F(\pi_0)$, of the measured parallaxes π_0, by the formula †

$$f(\pi)=F(\pi)-\frac{1}{4k'^2}F''(\pi)+\frac{1}{32k'^4}F''''(\pi)+\ldots \quad . \quad (20)$$

the frequency distribution of y in the group is given by $\Psi(y)$, where

$$\Psi(y)=\int_0^{\infty}\Phi(y)f(\pi)d\pi \quad . \quad . \quad . \quad . \quad (21)$$

together with nq stars at $+\infty$, where

$$q=\int_0^{\infty}pf(\pi)d\pi \quad . \quad . \quad . \quad . \quad (21a)$$

The value of the median absolute magnitude found from the calculated absolute magnitudes M_0 is therefore $M-y_1$, where

$$\int_{-\infty}^{y_1}\Psi(y)dy=\int_{y_1}^{\infty}\Psi(y)dy+q \quad . \quad . \quad . \quad (22)$$

But the mean true absolute magnitude is M. Hence the error in the median is y_1, where y_1 is given by (22).

If the accidental errors in the parallaxes were absent, *i.e.* if all the ξ's were zero, the median will obviously coincide with the mean true absolute magnitude. Again, if the true absolute magnitudes, instead of being dispersed about the value M, were all equal to M, *i.e.* if all the x's were zero, then also the median will coincide with M. To see this we consider that the absolute magnitude is a monotone function of the parallax when m is kept fixed. This makes the median absolute magnitude correspond to the median parallax for a sub-group of stars with a constant m. Hence for every such sub-group, and therefore also for the whole group, the median absolute magnitude is not affected by any systematic error.

It might appear at first sight that, since the median values of x and ξ are zero, the median value of y will also be zero, and hence that there will be no error in the median value of the absolute magnitudes M_0. But this in general is not so, for the frequency distribution of ξ is asymmetrical. In order to find what is the amount of systematic error in the median in practical cases, a typical example is worked out below.

References on next page

7. *A Numerical Example.*

We shall employ for this example the material used by Luyten. Consider the group of 27 G stars which have their H between 2·0 and 3·9. We shall first find the numerical value of h. For this purpose we take the spectroscopic absolute magnitudes and reduce each of them to the value 3·0 of H. Thus if for any particular star the spectroscopic absolute magnitude is M_1 and the value of H is H_1, we take

$$M'_1 = M_1 - 0{\cdot}65(H_1 - 3{\cdot}0) \qquad . \quad . \quad . \quad (23)$$

as the value of the spectroscopic absolute magnitude which this star would have had if its H had been 3·0. Here $+0{\cdot}65$ is the value of b in the formula (2) as given in Luyten's paper. The arithmetic mean of the absolute magnitudes thus obtained is $-0{\cdot}2$, and the standard deviation of these absolute magnitudes from the mean is 0·81. The accidental errors of the spectroscopic absolute magnitudes may be supposed to be normally distributed with a probable error of 0·4,‡ *i.e.* with a standard deviation of 0·59. Hence the standard deviation of x in equation (18) must be $\sqrt{\{(0{\cdot}81)^2 - (0{\cdot}59)^2\}} = 0{\cdot}56$, which gives $h = 1{\cdot}26$. The probable error of the trigonometrical parallaxes will be assumed to be $0''{\cdot}010$, giving $h' = 47{\cdot}7$.

Out of the 27 parallaxes 5 are negative, 7 are in the neighbourhood of $0''{\cdot}005$, 12 are in the neighbourhood of $0''{\cdot}025$, and 3 are greater than $0''{\cdot}050$. With the above value of the probable error 12 stars at $\pi = 0''{\cdot}005$ and 15 stars at $\pi = 0''{\cdot}025$ will give an observed distribution of parallaxes somewhat similar to the above, the number of negative parallaxes being precisely 5. For the sake of convenience, therefore, we assume that the true distribution of parallaxes is 12 values of $\pi = 0''{\cdot}005$ and 15 values of $\pi = 0''{\cdot}025$.

With the value of h derived above the frequency-distribution (9) gives ·340 n, ·245 n, ·075 n, and ·010 n stars between $M \pm 0{\cdot}25$ at $M = 0{\cdot}0$, $\pm 0{\cdot}5$, $\pm 1{\cdot}0$, $\pm 1{\cdot}5$, and $\pm 2{\cdot}0$ respectively. In view of the difficulty of evaluating the integrals occurring in (19), the simple assumption will here be made that there are ·340 n, ·245 n, ·075 n, and ·010 n stars for which x has the values 0·0, $\pm 0{\cdot}5$, $\pm 1{\cdot}0$ & $\pm 1{\cdot}5$ ~~and ±2·0~~ respectively. The distribution (17) for $\pi = 0''{\cdot}025$ requires 4·9, 11·6, 16·9, 16·9, 13·5, 9 7, 6·7, and 19·8 per cent. of the ξ's to lie between $-\infty$ and $-1{\cdot}5$, $-1{\cdot}25 \pm {\cdot}25$, $-0{\cdot}75 \pm {\cdot}25$, $-0{\cdot}25 \pm {\cdot}25$, $+0{\cdot}25 \pm {\cdot}25$, $+0{\cdot}75 \pm {\cdot}25$, $1{\cdot}25 \pm {\cdot}25$, and between $+1{\cdot}5$ and $+\infty$ respectively, the last group including the stars to which the absolute magnitude $-\infty$ is assigned on account of a negative parallax. To find $\Phi(y)$ at, say, $y = 0{\cdot}25$, we notice that the stars for which y has a value lying between $0{\cdot}25 \pm {\cdot}25$ consist of all the stars for which $x = -1{\cdot}5$ and ξ lies between $1{\cdot}75 \pm {\cdot}25$, or $x = -1{\cdot}0$ and ξ lies between $1{\cdot}25 \pm {\cdot}25$,

* Whittaker and Robinson, *The Calculus of Observations*, p. 170.

† *Ibid.*, p. 207.

‡ *Astroph. Journ.*, 53, 27.

. . . or $x = 1{\cdot}5$ and ξ lies between $-1{\cdot}25 \pm {\cdot}25$. Hence the number of such stars must be $n({\cdot}010 \times {\cdot}045 + {\cdot}075 \times {\cdot}067 + {\cdot}245 \times {\cdot}097 + \ldots + {\cdot}010 \times {\cdot}116)$, where $4{\cdot}5$ is the percentage of stars for which ξ lies between $1{\cdot}75 \pm {\cdot}25$. We can thus find $\phi(y)$, but since we want merely to know for how many stars y is positive, the numerical work is considerably shortened. We find thus that $0{\cdot}520\ n$ stars have a positive y and the rest have a negative y. If the median value of y were zero, the number of stars having a positive y would be not $0{\cdot}520\ n$, but $0{\cdot}500\ n$. The difference corresponds to an error in the median of $0^{\mathrm{M}}{\cdot}053$.

Similarly the error in the absolute magnitude of a group of stars, every one of which has a parallax of $0''{\cdot}005$, is found to be $0^{\mathrm{M}}{\cdot}071$.

Finally, the error in the median of the group of stars, 15 of which have a true parallax of $0''{\cdot}025$ and 12 a true parallax of $0''{\cdot}005$, is found to be $0^{\mathrm{M}}{\cdot}055$.

Formula (11) shows that the error in the absolute magnitude calculated from the mean parallax is $0^{\mathrm{M}}{\cdot}073$. Thus there is not much to choose between the two different methods on the score of the smallness of the systematic errors caused by the accidental errors in the parallaxes. But, as is well known, the probable error of the median is 25 per cent. greater than that of the arithmetic mean, and for this reason the use of formula (4) is preferable.

No correction was applied to the mean absolute magnitudes shown in Tables II. to X.

8. *Comparison with the Results of Van Rhijn.*

It is interesting to compare the relation between H and $\bar{M}$ arrived at here with that deducible from the mean parallaxes found by Van Rhijn * from an exhaustive study of measured parallaxes, proper motions, and radial velocities. Stars are selected for parallax measurement on account of their brightness or large proper motion, and the influence of this selection on the relation between H and $\bar{M}$ is also clearly brought out by such a comparison. The last sections of tables xxxiv. to xxxviii. of Van Rhijn give the mean parallaxes of stars of various spectral types of determined proper motion and apparent magnitude. From every one of these we can construct another table giving the mean absolute magnitude when H is known. The Tables XI. to XV. have thus been obtained, the entries being mean absolute magnitudes and the arguments H and m. They show that the mean absolute magnitude is not a one-valued function of H. To know the mean absolute magnitude definitely we must know also the apparent magnitude (or the proper motion).

Reference on next page.

TABLE XI.

Mean Absolute Magnitudes of A Stars of Determined* H *and* m, *calculated from Table* 34 *of Van Rhijn.*

H. \ m	2.	3.	4.	5.	6.	7.	8.	9.	10.
−3	−2·4								
−2	−2·1	−1·9							
−1	−1·6	−1·6	−1·4						
0	−1·0	−1·1	−1·1	−0·9					
1	−0·4	−0·6	−0·7	−0·5	−0·3				
2	0·2	−0·1	−0·2	−0·2	−0·1	0·2			
3	0·9	0·5	0·3	0·2	0·3	0·4	0·7		
4	1·6	1·2	0·8	0·7	0·6	0·7	0·9	1·2	
5	2·3	1·8	1·5	1·2	1·0	1·0	1·2	1·4	1·7
6	3·0	2·5	2·0	1·7	1·5	1·4	1·5	1·7	1·9
7	3·7	3·1	2·7	2·2	1·9	1·8	1·8	1·9	2·1
8		3·8	3·3	2·8	2·5	2·2	2·2	2·2	2·3
9			3·9	3·4	3·0	2·7	2·5	2·5	2·6
10				4·0	3·5	3·2	3·0	2·9	2·8
11					4·1	3·7	3·4	3·3	3·1
12						4·2	3·9	3·6	3·5
13							4·3	4·0	3·9
14								4·4	4·2
15									4·6

TABLE XII.

Mean Absolute Magnitude of F Stars of Determined H *and* m, *calculated from Table* 35 *of Van Rhijn.*

H. \ m	2.	3.	4.	5.	6.	7.	8.	9.	10.
−3	−3·1								
−2	−2·8	−2·4							
−1	−2·4	−2·1	−1·7						
0	−2·0	−1·7	−1·4	−0·9					
1	−1·4	−1·2	−1·0	−0·6	−0·2				
2	−0·7	−0·6	−0·4	−0·2	0·2	0·5			
3	0·1	0·1	0·2	0·3	0·5	0·9	1·2		
4	1·0	0·8	0·8	0·9	1·0	1·3	1·6	1·8	
5	1·9	1·6	1·5	1·5	1·6	1·8	2·0	2·3	2·4
6	2·7	2·4	2·2	2·2	2·2	2·3	2·5	2·7	2·9
7	3·7	3·2	3·0	2·8	2·7	2·8	2·9	3·2	3·4
8	4·6	4·1	3·7	3·5	3·4	3·4	3·5	3·6	3·8
9	5·5	4·9	4·5	4·2	4·0	3·9	3·9	4·1	4·3
10	6·3	5·7	5·2	4·9	4·7	4·5	4·5	4·6	4·7
11		6·5	5·9	5·6	5·3	5·1	5·1	5·0	5·2
12			6·7	6·2	5·9	5·7	5·6	5·5	5·6
13				6·9	6·5	6·2	6·1	6·0	6·0
14					7·1	6·8	6·6	6·5	6·5
15						7·3	7·1	7·0	6·9
16							7·6	7·4	7·3
17								7·8	7·7
18									8·1

* "On the Mean Parallaxes of Stars of determined Proper Motion, Apparent Magnitude, and Galactic Latitude for each Spectral Class," *Groningen Publications*, No. 34, 1923.

Table XIII.

Mean Absolute Magnitude of G Stars of Determined H *and* m, *calculated from Table* 36 *of Van Rhijn.*

H. \ m	2.	3.	4.	5.	6.	7.	8.	9.	10.
-3	-3.1								
-2	-2.9	-2.4							
-1	-2.6	-2.8	-1.7						
0	-2.1	-1.8	-1.4	-0.9					
1	-1.5	-1.3	-1.0	-0.6	-0.2				
2	-0.8	-0.7	-0.4	-0.2	0.2	0.5			
3	0.0	0.0	0.1	0.3	0.6	0.9	1.2		
4	0.8	0.7	0.7	0.9	1.1	1.3	1.7	1.8	
5	1.7	1.5	1.4	1.5	1.6	1.8	2.1	2.3	2.5
6	2.6	2.3	2.1	2.2	2.2	2.4	2.5	2.8	3.0
7	3.5	3.1	2.9	2.8	2.8	2.9	3.1	3.3	3.5
8	4.4	3.9	3.6	3.5	3.4	3.5	3.6	3.8	4.0
9	5.3	4.8	4.4	4.2	4.1	4.1	4.1	4.3	4.5
10	6.2	5.5	5.2	4.9	4.7	4.7	4.6	4.7	4.9
11		6.4	5.9	5.6	5.3	5.2	5.2	5.3	5.4
12			6.6	6.2	5.9	5.8	5.7	5.8	5.9
13				6.9	6.5	6.3	6.3	6.2	6.3
14					7.2	6.9	6.8	6.7	6.7
15						7.4	7.2	7.2	7.2
16							7.7	7.6	7.5
17								8.0	8.0
18									8.4

Table XIV.

Mean Absolute Magnitude of K Stars of Determined H *and* m, *calculated from Table* 37 *of Van Rhijn.*

H. \ m	2.	3.	4.	5.	6.	7.	8.	9.	10.
-3	-3.0								
-2	-2.8	-2.4							
-1	-2.4	-2.2	-1.8						
0	-2.0	-1.8	-1.5	-1.1					
1	-1.5	-1.3	-1.2	-0.9	-0.5				
2	-0.9	-0.8	-0.7	-0.5	-0.2	0.1			
3	-0.2	-0.2	-0.1	0.0	0.2	0.4	0.7		
4	0.5	0.5	0.5	0.6	0.7	0.8	1.1	1.4	
5	1.3	1.2	1.2	1.3	1.3	1.4	1.5	1.7	2.0
6	2.0	2.0	2.0	2.0	1.9	2.0	2.1	2.2	2.4
7	2.8	2.8	2.8	2.7	2.7	2.7	2.7	2.8	2.8
8	3.6	3.6	3.5	3.5	3.5	3.4	3.4	3.4	3.5
9	4.4	4.4	4.4	4.3	4.3	4.2	4.2	4.1	4.1
10	5.1	5.1	5.1	5.1	5.0	5.0	4.9	4.9	4.8
11		5.9	5.9	5.9	5.8	5.8	5.8	5.7	5.6
12			6.7	6.6	6.6	6.6	6.5	6.5	6.4
13				7.4	7.4	7.4	7.3	7.3	7.2
14					8.2	8.2	8.1	8.1	8.1
15						8.9	8.9	8.9	8.9
16							9.7	9.7	9.6
17								10.4	10.4
18									11.2

TABLE XV.

Mean Absolute Magnitude of M Stars of Determined H *and* m, *calculated from Table* 38 *of Van Rhijn.*

H. \ m	2.	3.	4.	5.	6.	7.	8.	9.	10.
− 3	−2·1								
− 2	−1·8	−1·8							
− 1	−1·6	−1·5	−1·5						
0	−1·2	−1·2	−1·2	−1·1					
1	−0·9	−0·9	−0·9	−0·9	−0·8				
2	−0·5	−0·6	−0·6	−0·6	−0·5	−0·5			
3	−0·2	−0·2	−0·2	−0·2	−0·2	−0·2	−0·2		
4	0·2	0·2	0·1	0·1	0·1	0·2	0·2	0·2	
5	0·5	0·5	0·5	0·5	0·5	0·5	0·5	0·4	0·5
6		0·9	0·9	0·9	0·9	0·8	0·8	0·8	0·7
7			1·2	1·2	1·2	1·2	1·2	1·2	1·2
8				1·6	1·6	1·6	1·5	1·5	1·5
9					2·0	1·9	1·9	1·9	1·9
10						2·4	2·3	2·2	2·2
11							2·7	2·7	2·6
12								3·0	3·0
13									3·5

These tables show that for the smaller values of H the range in $\bar{M}$ for a given value of H is in most cases considerable. But several of the entries in these tables have no meaning in practice. Consider for example Table XIII. for the G stars. There are in the sky only two G stars brighter than apparent magnitude 2·4. Hence entries under $m=2$ can be neglected. Again, the blank space in the lower left-hand corner represents the omission of stars with $\mu > 4''\cdot00$ per year. The lowest entries correspond to $\mu = 4''\cdot00$ and the entries immediately above them to $\mu = 2''\cdot50$. On account of the scarcity of stars with such large motions we may neglect these entries also. A few more of the mean absolute magnitudes for the bright stars with very large proper motions may similarly be neglected. Only two G stars with $m > 9\cdot4$ are included in the present discussion, and therefore we may neglect the entries under $m = 10$. The blank space in the upper right-hand corner represents the omission of stars with $\mu < 0''.010$. The top entries correspond to stars with $\mu = 0''.010$. These also can be neglected. The extreme curves showing the relation between H and $\bar{M}$ as given by the remaining entries in the table are shown in fig. 3 by the dotted lines. We see that our curve lies between the two extreme curves obtained from Van Rhijn's results. The curve on the right corresponds to the fainter stars ($m = 5$ to 9) of small proper motion ($\mu = 0''.020$) up to the point $H = 5$, $M = 2.3$, and to stars of apparent magnitude 9 only beyond this point. We see that as more and more parallaxes of the fainter stars with small proper motions are measured, the lower part of the curve giving the relation between H and $\bar{M}$ derived from all the measured parallaxes will shift more and more to the right.

The agreement in the case of the G stars between the line obtained in this paper and the line obtained from Van Rhijn's results is exceptional. In the case of the A, F, and K stars the lines of Van Rhijn are more to the left for those portions of the figure which correspond to the smaller values of M and therefore to the smaller values of π. This can be seen from figs. 1, 2, and 4, the dotted lines denoting the lines of Van Rhijn. This indicates that the smaller trigonometrical parallaxes are on the average relatively larger than the mean parallax derived from parallactic motions and similar considerations, a point which has been noted by Van Rhijn * also. The disagreement between the line obtained in this paper for the K5 and M giants and Van Rhijn's line for the M giants has been mentioned before. The line for the M stars alone agrees fairly well with Van Rhijn's line, but, unlike the case of the A, F, and K stars, the smaller trigonometrical parallaxes in this case are slightly smaller than the mean parallaxes of Van Rhijn. The disagreement in the lines for the K stars is very marked. In order to see if the inclusion of the K5 stars among the K stars will produce better agreement, a solution was made for all the K0, K2, and K5 stars together. The line thus obtained practically coincides with the line for the K0 and K2 stars. The disagreement obviously is not due to this cause. The absolute magnitudes of the M stars with a value of H greater than 10, as obtained from Van Rhijn's parallaxes, is quite different from the absolute magnitudes obtained here. Evidently his parallaxes do not apply to the dwarfs.

In conclusion, I wish to express my gratitude to Professor Sampson for his kind help and encouragement.

Reference on next page.

Summary.

1. The relation between $H \equiv 5 + m + 5 \log \mu$ and mean absolute magnitude has been found for each spectral type separately. The result is shown in Table I.

2. The discussion is based upon the trigonometrical parallaxes, The negative parallaxes have not been rejected.

3. The mean absolute magnitude has been calculated from the mean parallax. The advantage of this procedure, the systematic error to which it gives rise, and the effect of the accidental errors of the parallaxes and proper motions are considered.

4. The relation between H and mean absolute magnitude is found to be linear except for the F and G stars. In the case of the K and M stars giants and dwarfs had to be considered separately.

5. Comparison with the mean parallaxes published by Van Rhijn shows agreement in the case of G stars. In the case of A, F, and K stars the smaller trigonometrical parallaxes are in the mean greater than the mean parallaxes of Van Rhijn. In the case of M stars there is fair agreement, the mean of the smaller trigonometrical parallaxes being slightly less than the corresponding mean parallaxes of Van Rhijn.

Royal Observatory, Edinburgh:
1924 *December* 5.

* *Loc. cit.*, p. 61.

On the Progression of Stellar Velocity with Absolute Magnitude.

On the Progression of Stellar Velocity with Absolute Magnitude.

By GORAKH PRASAD

It is well known that there is a correlation between the velocities and absolute magnitudes of stars, but the determination of the rate of change of velocity with absolute magnitude presents unusual difficulties. Eddington and Miss Douglas* have shown that the systematic error in this quantity produced by the accidental errors of the magnitudes is very great. Thus the value of the increase of the cross-component (i.e., at right angles to the direction towards the solar apex) of the linear tangential velocity per unit absolute magnitude for the K giants has been shown by them to be 3·46 or 7·54 km. per sec. according as the probable error of the absolute magnitudes is taken to be $0^{M}.3$ or $0^{M}.4$. The value $0^{M}.5$ for the probable error is not unlikely. In fact the values of the probable

---

* Monthly Notices, 83 (1923), 112

errors for various groups of the stars considered by Eddington and Miss Douglas as given by Adams and his collaborators* range from $0^{M}.53$ to $0^{M}.32$. From the material given by Eddington and Miss Douglas the value of the increase of linear cross-velocity per unit absolute magnitude may be calculated to be 22·5 km. per sec. We see how sensitive this quantity is to a change in the adopted value of the probable error of the absolute magnitudes.

If the stars are grouped according to any characteristic in the value of which the absolute magnitudes, or the parallaxes derived from them, do not enter, the mean absolute magnitude for the group will have a far smaller accidental error and therefore the value of the increase of linear cross-velocity per unit absolute magnitude will be more reliable. The function $H \equiv m + 5 + 5 \log \mu$, where μ is the total proper motion, is a suitable characteristic for grouping the stars, since there is a relation between H and mean absolute magnitude and

---

* Ap. J., 53 (1921), 27.

since in the calculation of H neither the absolute magnitude nor the parallax is used.

The investigation of the relation between mean absolute magnitude and mean linear cross-velocity when the stars are grouped according to their value of H is the main object of this paper. The relation between mean absolute magnitude and mean total tangential velocity when the stars are grouped according to H can be immediately derived from the relation between H and mean absolute magnitude found in a previous paper of mine*. This will be taken up first.

1. The rate of increase of total tangential linear velocity with absolute magnitude deduced from the relation between H and $\overline{M}$.

Let the stars be grouped according to the value of their H, the range of the values of H within any one group being small. Consider any one group and denote the mean of the values of H by H and the mean of the absolute mag-

---

* The first paper in this Thesis.

nitudes by $\overline{M}$. Let the true (i.e., unaffected by accidental errors) absolute magnitudes of the individual stars be denoted by M'. Put

$$\overline{M} = M' + x. \qquad \ldots \ldots \quad (1)$$

We suppose that x has a normal frequency distribution, the standard deviation being σ. Let the relation between H and $\overline{M}$ be

$$\overline{M} = a + bH + cH^2. \qquad \ldots \ldots \quad (2)$$

Since c is small, this gives approximately

$$H = 5A + (5B+1)\overline{M} + 5C\overline{M}^2, \quad \ldots \quad (3)$$

where
$$A = -\frac{1}{5}\left(\frac{a}{b} + \frac{a^2 c}{b^3}\right),$$

$$B = -\frac{1}{5}\left(1 - \frac{1}{b} - \frac{2ac}{b^3}\right),$$

and
$$C = -\frac{1}{5}\frac{c}{b^3}.$$

Bearing in mind that $H = m + 5 + 5\log\mu$

and $M' = m + 5 + 5\log\pi$,

we get from (3) and (1)

$$v_\mu = 10^{A + B\overline{M} + C\overline{M}^2} \cdot 10^{0.2x}, \quad \ldots \quad (4)$$

where $v_\mu = \mu/\pi$ and is the total linear tangential velocity, its unit being one astronomical unit per annum, or 4·74 km. per sec. Expanding $10^{0.2x}$ in powers of x and taking the mean, we find that the mean velocity

$$\bar{v}_\mu = 10^{A + B\bar{M} + C\bar{M}^2}(1 + 0{\cdot}106\,\sigma^2), \quad . \quad . \quad . \quad (5)$$

powers of σ higher than the third being neglected. We have therefore

$$\log \bar{v}_\mu = A + B\bar{M} + C\bar{M}^2 + \log(1 + 0{\cdot}106\,\sigma^2). \quad . \quad (6)$$

In cases in which B is small, we can express $\bar{v}_\mu$ in the form

$$\bar{v}_\mu = a' + b'\bar{M} + c'\bar{M}^2 \quad . \quad . \quad . \quad (7)$$

by expanding $10^{B\bar{M} + C\bar{M}^2}$ in powers of $\bar{M} - M_1$, where M_1 is some convenient number nearly equal to the mean absolute magnitude of all the stars.

The case in which the relation between H and $\bar{M}$ is a straight line can be obtained from the above by putting $c = C = 0$. The values of A, B, C for various spectral types as calculated from the values of a, b, c given in Table I* of my previous paper are shown below, v_μ being supposed to be measured in km. per sec. The term $\log(1 + 0{\cdot}106\,\sigma^2)$ is generally very small.

[Table

---

* See p. 5 of this Thesis.

Table I.

Values of A, B, C in formula (6) for various spectral classes, v_n being measured in km. per sec.

Spectral Class	Range of the values of M	A	B	C
B8, B9, A0, A2, A3, A5	0·2 to 2·5	0·89	0·291	
F0, F2, F5, F8	0·2 to 5·1	0·93	0·081	0·0101
F0, F2, F5, F8	2·9 to 5·1	0·18	0·360	
G0, G5	−1·0 to 6·2	1·17	0·042	0·0057
G0, G5	3·7 to 6·2	0·63	0·226	
K0, K2 Giants	−0·1 to 1·6	0·91	0·451	
K0, K2, K5 Giants	−0·3 to 1·8	1·00	0·349	
K0, K2 Dwarfs	3·4 to 8·4	1·50	0·031	
K0, K2 Dwarfs	1·6 to 8·4	1·57	0·021	
K5, Ma, Mb, Mc Giants	−2·1 to 2·1	1·25	0·109	
Ma, Mb, Mc Giants	−1·3 to 1·1	1·33	0·260	
K5, Ma, Mb, Mc Dwarfs	7·8 to 10·5	0·49	0·133	

2. The increase of linear cross-velocity with absolute magnitude.

The above gives an increase of about 21 km. per sec. per unit absolute magnitude for the K giants. This is very high. In order to get results which could be compared with those of Eddington and

Miss Douglas, the increase of linear cross-velocity with absolute magnitude was calculated for the stars considered by these authors, viz., the stars from type G8 to K2 inclusive (Mt. Wilson estimated spectrum) brighter than absolute magnitude $3^M.5$ contained in the Astrophysical Journal, 53 (1921), p.13, excluding such stars as are not included in Boss's Preliminary General Catalogue.

The proper motions were taken from Boss. For calculating the cross-component of the proper motion, the position angle of the solar ant-apex was read off from the charts published by Smart* in which the position of the ant-apex is taken to be R.A. $6^h\ 0^m$, Dec. $-34°$. The linear cross-velocity v was obtained by dividing the cross-component of the proper motion by the spectroscopic parallax. The stars, 285 in number, were arranged in order of H and divided into five groups, each containing the same number of stars. The mean

---

* "Charts giving the Angular Distances of Stars from, and the Position Angles relative to the Ant-apex of the Solar Motion", by W. M. Smart. London; Royal Astronomical Society, 1923. See also Monthly Notices, 83 (1923), 465.

values of H, M and v are shown in Table II., the unit of v being a velocity of 4·74 km. per sec. In order to reduce the accidental effect of a few excessive motions the table also contains the mean values of v when (1) the three largest velocities in each group and (2) the six largest velocities in each group are omitted. Correcting for the effect of the omission of the largest velocities in exactly the same way as has been

Table II.

Mean values of H, M and v.

Number	$\overline{H}$	$\overline{M}$	$\overline{v}$(all)	$\overline{v}$(-3)	$\overline{v}$(-6)
1	-0·09	0·46	0·73	0·45	0·39
2	2·42	0·96	1·50	1·16	1·05
3	4·07	1·19	2·62	2·39	2·20
4	5·23	1·07	4·14	3·82	3·57
5	7·00	1·64	5·69	5·00	4·63

done by Eddington and Miss Douglas, the formulae giving $\overline{v}$ in terms of $\overline{M}$ are

$$\overline{v} = -7{\cdot}1 + 19{\cdot}8\,\overline{M} \text{ km. per sec. (all stars)} \quad . \quad . \quad . \quad (8)$$

$$\overline{v} = -8{\cdot}2 + 20{\cdot}4\,\overline{M} \text{ km. per sec. (15 rejected)} \quad . \quad . \quad . \quad (9)$$

$$\overline{v} = -8{\cdot}4 + 20{\cdot}8\,\overline{M} \text{ km. per sec. (30 rejected)} \quad . \quad . \quad (10)$$

the unit of v being now one km. per sec. The three formulae agree very nearly. Taking the middle one as definitive, we find an increase of 20·4 km. per sec. per unit absolute magnitude. This may be compared with the values 3·5, 7·5 and 22·5 km. per sec. of Eddington and Miss Douglas mentioned before and the values 3·2 km. per sec. and 1·4 km. per sec. of Adams, Strömberg and Joy* for the increase per unit absolute magnitude of the total space velocity and radial velocity respectively. It should be mentioned that in the latter investigation the effect of the accidental errors of the absolute magnitudes is not allowed for.

The probable error of the coefficient of $\overline{M}$ in equation (9) is ± 4·2 km. per sec., which is 21 per cent. of the quantity sought. Thus the present determination is of the same order of accuracy as that of Eddington and Miss Douglas; the probable error in their case is 19 per cent. when it is assumed that the value of the probable error of the absolute magnitudes is accurately known to be $0^{M}·3$.

---

* Ap. J., 54 (1921), 9.

The points for the five groups and the straight line given by equation 9 are shown in figure 1.

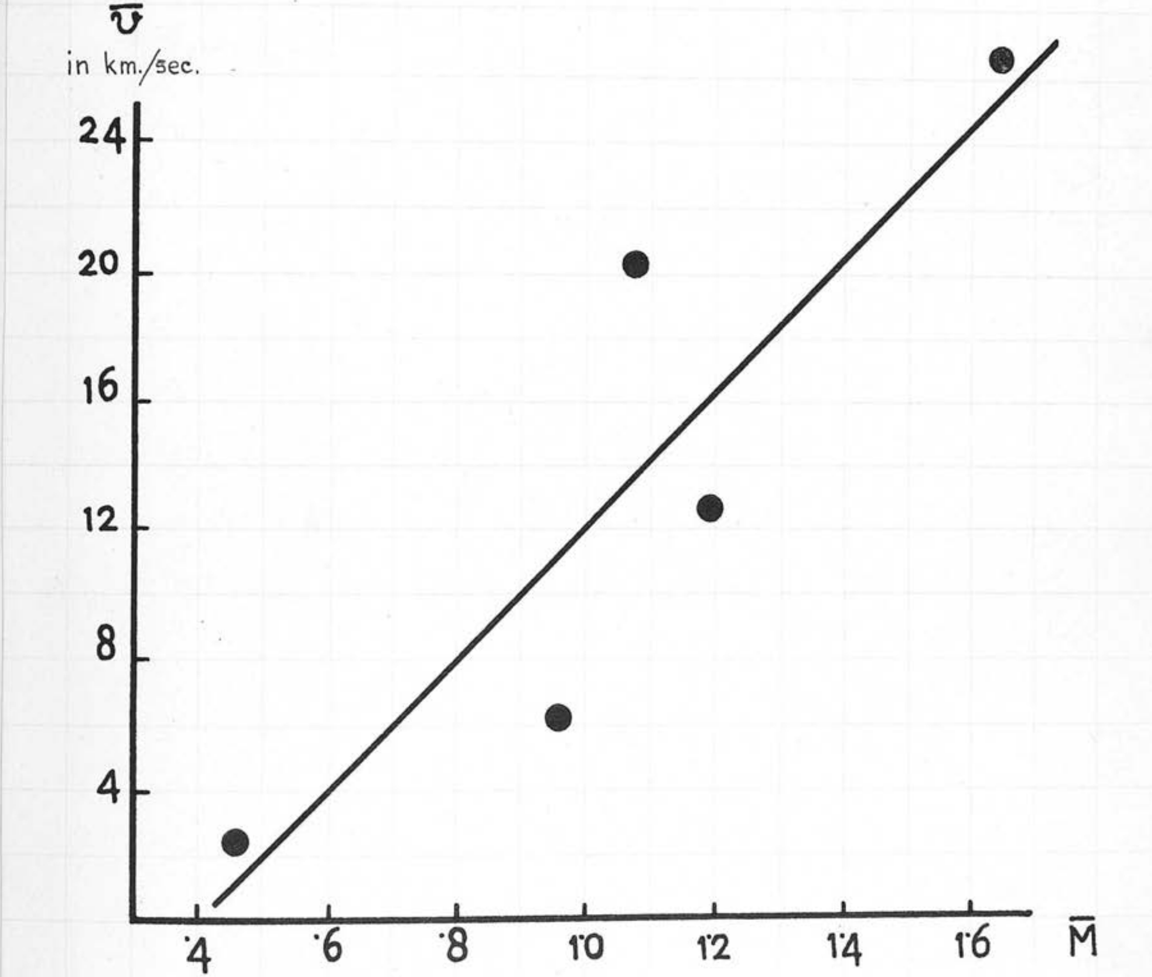

Fig. 1. — Relation between $\overline{M}$ and $\overline{v}$.

There are two factors tending towards a high value of the rate of change of linear cross-velocity with absolute magnitude. Firstly, when the groups are formed according to H and the mean of the absolute magnitudes is taken, instead of the groups being formed according to the absolute magnitude itself, the apparent dispersion of the absolute magnitudes caused by the accidental errors is almost eliminated. The total range of the absolute magnitudes being thus diminished the rate of increase of velocity with absolute magnitude comes out as high. Secondly, there is not perfect correlation between linear cross-velocity and absolute magnitude and, as is well known*, in cases of imperfect correlation the straight lines obtained according as the groups are formed with respect to one or the other of the two characteristics between which a relation has to be found have different slopes, and the weaker the correlation between the two characteristics, the greater is the difference in their slopes. Apparently, grouping the stars according to H has an effect similar to grouping them with respect to v. An attempt to find the increase of velocity with absolute magnitude by grouping the stars with respect to v seems hopeless, because v is much more uncertain than H and because the use of the parallax derived from

*See, for example, Brunt, The Combination of Observations, Cambridge, 1917, p.155

the absolute magnitude in calculating it further complicates the problem. The masking of the progression of velocity with absolute magnitude on account of the accidental errors in M and v is well shown by the "scatter diagram" (figure 2), in which the points corresponding to individual stars are plotted. One point $(M = -0{\cdot}1, v = 24{\cdot}7)$ lies outside the figure.

We see that <u>when the stars are grouped according to M, the increase of the cross-component of the linear tangential velocity has a much higher value than when they are grouped according to the observed absolute magnitude. Making an allowance for the effect of the accidental errors of the absolute magnitudes brings the two results in closer agreement. For complete agreement, the value of the probable error of the absolute magnitudes should be a little less than $0^{M}{\cdot}5$</u>, a value for or against which there is little evidence.

On account of the very imperfect correlation between linear cross-velocity and absolute magnitude little reliance should be placed on the mean

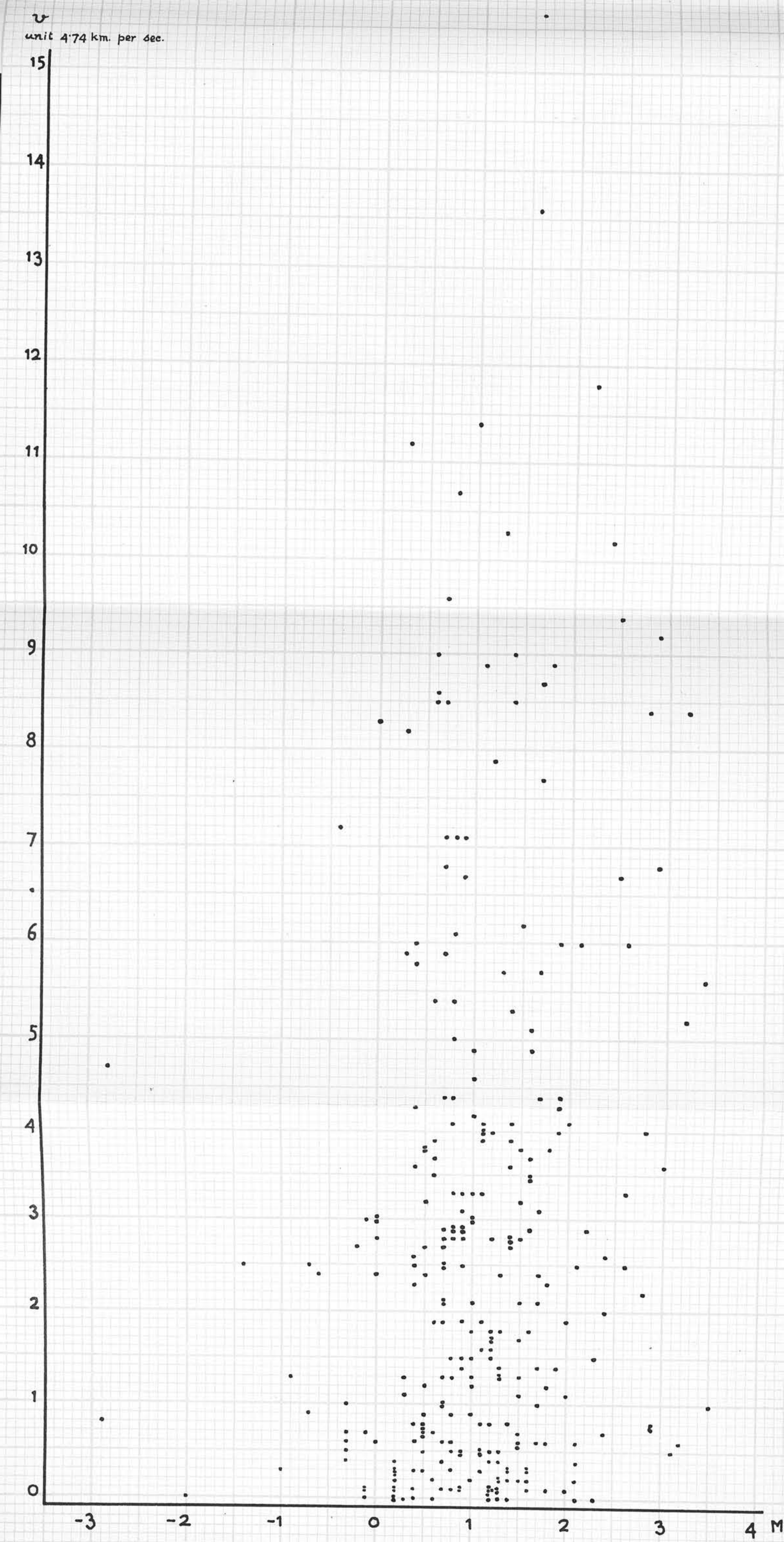

Fig. 2.—"Scatter Diagram" of M and υ

linear cross-velocity for a small group of stars calculated from the numerical formulae here derived.

In conclusion, I wish to express my gratitude to Professor Sampson for his kind interest in the preparation of this paper.

Royal Observatory: Edinburgh.

January, 1925.

MEAN ABSOLUTE MAGNITUDE OF A GROUP OF STARS: NOTE ON A PAPER BY MESSRS. YOUNG AND HARPER. By GORAKH PRASAD, M.Sc.

Monthly Notices of R.A.S., May 1924.

Reprinted from the Monthly Notices of the Royal Astronomical Society, *Vol. LXXXIV. No. 7.*

MEAN ABSOLUTE MAGNITUDE OF A GROUP OF STARS: NOTE ON A PAPER BY MESSRS. YOUNG AND HARPER.

BY

GORAKH PRASAD, M.Sc.

(*Communicated by the Astronomer Royal for Scotland.*)

The absolute magnitudes and parallaxes, together with the methods employed for finding them, for stars of types F to M have been published by the Mount Wilson,* the Norman Lockyer, † and the Dominion Astrophysical ‡ Observatories. The method employed in the last paper for establishing the curves relating intensity differences of pairs of lines in the spectra of stars with absolute magnitude differs from the methods employed in the other two papers. One is struck by the elaborateness of the procedure adopted by Young and Harper and the comparative poorness of the result obtained. A casual glance at the sample figure § given by these authors with those given by Rimmer || suffices to demonstrate the far greater reliability of the latter curves. The plotted points in the figure of Young and Harper are so scattered that drawing a curve through them is highly arbitrary. No doubt a part of this scattering is due to the use of groups of stars instead of individual stars, and the procedure employed to make up for the consequent lack of points; but, as will appear from this note, some of it is due to the use of unsuitable formulæ for calculating mean absolute magnitudes. The correction found by the authors for the mean absolute magnitude computed from the trigonometrical parallaxes by their first method is

* W. S. Adams, A. H. Joy, G. Strömberg, and Cora G. Burwell, "The Parallaxes of 1646 Stars derived by the Spectroscopic Method," *Contributions from the Mount Wilson Observatory*, No. 199, and *Astrophys. J.*, 53, 1921.

† W. B. Rimmer, "The Luminosities and Parallaxes of 500 Stars, Types F0 to Mb," *Memoirs of the Royal Astronomical Soc.*, 62, 1923.

‡ R. K. Young and W. E. Harper, "The Absolute Magnitudes and Parallaxes of 1080 Stars," *Journal of the Royal Astr. Soc. of Canada*, 18, Nos. 1–2, 1924. (The complete results are not yet out.)

§ *Loc. cit.*, p. 29, fig. 8. It will be noted that the curve itself has not been shown in the figure. A curve drawn through the plotted points is described as a preliminary curve; but this does not minimise the importance of what is said here, as all the uncertainty of this curve enters into the final one.

|| *Loc. cit.*, p. 118, figs. 2 and 3.

wrong, and it is fortunate that the correction was not applied. The second method used by the authors introduces more serious errors.

1. *Correction due to the Accidental Errors in the Trigonometrical Parallaxes.*

Young and Harper have shown * that if the absolute magnitude of every star of a group is computed from its trigonometrical parallax and the mean of these absolute magnitudes is taken, the correction to the computed mean absolute magnitude due to accidental errors in the trigonometrical parallaxes is

$$-\frac{5h}{c^{1/2}}\int_0^\infty \frac{\log(1+x/\pi)+\log(1-x/\pi)}{10^{h^2x^2(0\cdot4343)}}dx, \quad . \quad . \quad . \quad (1)$$

where h is the modulus of precision of the trigonometrical parallaxes, c is the ratio of the circumference of a circle to its diameter, and π is the true parallax of the group of stars, if all the stars have the same parallax. The formula is stated to hold for a group of stars which do not have the same π and m, but it is not stated what meaning is to be attached to π in the expression (1) in this case. Apart from this, for values of x greater than π the integrand in (1) becomes imaginary, and for this reason alone (1) will cease to be applicable. In the most favourable case, when all the stars of the group have the same parallax π, which is so large that there is little likelihood of an accidental error being greater than it, we can replace (1) by

$$-\frac{5h}{c^{1/2}}\int_0^a \frac{\log(1+x/\pi)+\log(1-x/\pi)}{10^{h^2x^2(0\cdot4343)}}dx, \; 0<a<\pi,$$

and this objection disappears, but even in this case the formula is not applicable. The reason is this. The formula is based on the assumption that the stars of the group have the same true absolute magnitude M and the same π. The observed values of π, it is stated, will cluster about π, some being less and some greater. Now the stars are grouped together *according to the absolute magnitude computed from the observed* π. Hence a star whose observed parallax is appreciably different from π is put under a different group altogether. A little consideration will show that it is essential for the validity of the formula (1) that the stars of the group have the same *true* absolute magnitude. Since we have no means of knowing these, and all groups are necessarily based upon absolute magnitudes affected with errors caused by the errors in the trigonometrical parallaxes, formula (1) is of no practical use.

We can find as follows the excess of the true mean absolute magnitude over the computed mean absolute magnitude caused by the accidental errors in the trigonometrical parallaxes.†

Let the total number of stars whose trigonometrical parallaxes π_0 have been measured be N, where N is large. Assuming that the

* *Loc. cit.*, p. 38.

† The investigation by F. H. Seares, "A Troublesome Systematic Error" (*Monthly Notices*, 84, 15, 1923), is not applicable to the present case.

frequency distribution of these stars as regards parallax is continuous, we can suppose that the number of stars found to have a measured parallax between π and $\pi + d\pi$ is $\mathrm{N}y(\pi)d\pi$. In consequence of the accidental errors in the measured parallaxes, the number, N_1, of stars having a true parallax between π and $\pi + d\pi$ is not $\mathrm{N}y(\pi)d\pi$ but $\mathrm{N}u(\pi)d\pi$, where

$$u(\pi) = y(\pi) - \frac{1}{4h^2}y''(\pi) + \frac{1}{32h^4}y^{\mathrm{iv}}(\pi) + \ . \ . \ . ,$$

h being the modulus of precision of the measured parallaxes.*

On account of the accidental errors, the N_1 stars with a true parallax between π and $\pi + d\pi$ contribute a number, N_2, to the group of stars with a measured parallax between π_1 and $\pi_1 + d\pi_1$, where

$$\mathrm{N}_2 = \mathrm{N}u(\pi)d\pi h c^{-1/2} e^{-h^2(\pi - \pi_1)^2} d\pi_1.$$

The error in the measured parallax of these N_2 stars is $\pi - \pi_1$ (in the sense true—measured). Out of the N stars, let the total number of stars having a measured parallax (say π_1') between π_1 and $\pi_1 + d\pi_1$ be N_3. The number of such stars out of these N_3 stars as have a spectral type S and an apparent magnitude m satisfying the inequality

$$\mathrm{M} \leqq m + 5 + 5 \log_{10}\pi_1', < \mathrm{M} + d\mathrm{M}$$

will obviously depend on π_1, S, and M. Hence we can assume for the number of such stars the expression

$$\mathrm{N}_3 f(\pi_1, \mathrm{S}, \mathrm{M})d\mathrm{M}.$$

Since the absolute magnitude, defined as the apparent magnitude at a distance corresponding to a parallax of 0″·1 (10 parsecs) is given by

$$\mathrm{M} = m + 5 + 5 \log_{10}\pi, \quad . \quad . \quad . \quad . \quad (2)$$

these stars have a computed absolute magnitude between M and $\mathrm{M} + d\mathrm{M}$. Hence out of the N_2 stars the number of such stars as have a spectral type S and a computed absolute magnitude between M and $\mathrm{M} + d\mathrm{M}$ is N_4, where

$$\mathrm{N}_4 = \mathrm{N}_2 f(\pi_1, \mathrm{S}, \mathrm{M})d\mathrm{M}.$$

Differentiating equation (2), we see that an error $\pi - \pi_1$ in the parallax produces an error $k(\pi - \pi_1)/\pi$ in the absolute magnitude, where $k = 2{\cdot}17$. This expression is valid only when $\pi - \pi_1$ is small compared with π. In general the error is

$$5(\log \pi - \log \pi_1).$$

The error in the sum, Σ M, of the computed absolute magnitudes due to errors in the parallaxes of the N_4 stars is therefore

$$5\mathrm{N}_4(\log \pi - \log \pi_1).$$

* Eddington, *Monthly Notices*, **73**, 359, 1913, or Whittaker and Robinson, *The Calculus of Observations*, p. 207.

The total error in Σ M, obtained by integrating this, is

$$5\mathrm{N}hc^{-1/2}d\mathrm{M}\int_a^\infty\int_0^\infty u(\pi)e^{-h^2(\pi-\pi_1)^2}f(\pi_1, \mathrm{S}, \mathrm{M})(\log\pi-\log\pi_1)d\pi d\pi_1,$$

where a is the smallest parallax used for the computation of an absolute magnitude.

The error in the mean absolute magnitude is therefore

$$\frac{5\mathrm{N}hc^{-1/2}d\mathrm{M}\int_a^\infty\int_0^\infty u(\pi)e^{-h^2(\pi-\pi_1)^2}f(\pi_1, \mathrm{S}, \mathrm{M})(\log\pi-\log\pi_1)d\pi d\pi_1}{\mathrm{N}hc^{-1/2}d\mathrm{M}\int_a^\infty\int_0^\infty u(\pi)e^{-h^2(\pi-\pi_1)^2}f(\pi_1, \mathrm{S}, \mathrm{M})d\pi d\pi_1}. \quad (3)$$

Since $u(\pi)$ is zero in the neighbourhood of $\pi=0$ and $\pi=\infty$, the range of integration 0 to ∞ may be replaced by a to β; $a>0$, β finite. The sign of the correction will depend upon the functions u and f. If we assume for the moment that the variation of $f(\pi_1, \mathrm{S}, \mathrm{M})$ with π_1 is negligible, *i.e.* that there is approximately the same percentage of stars of spectral type S and absolute magnitude M in every group of stars formed according to the measured parallax, we see that the expression (3) is zero if $u(\pi)=0$ for $\pi<a$, and is some constant for other values of π. This shows that the correction chiefly arises from the unequal frequency distribution of the stars as regards parallax. Generally, $u(\pi)$ increases as π diminishes. This, it is easy to see, will make the integral in the numerator of (3) negative. The computed mean absolute magnitude will thus be greater than the true value, or in other words, the stars will be computed more dwarfish than they really are. The effect of the variation of a and $f(\pi_1, \mathrm{S}, \mathrm{M})$ can now be considered. An increase in a, that is, leaving out some more of the small parallaxes will make the correction increase by a negative amount. If $u(\pi)$ is a constant, a decrease of $f(\pi_1, \mathrm{S}, \mathrm{M})$ as π_1 increases will make the correction positive, an increase of $f(\pi_1, \mathrm{S}, \mathrm{M})$ as π_1 increases will make it negative. Hence, in general, for a group of stars in which the same proportion of large and small parallaxes occur as in the complete list, the correction will be negative. The correction for a group of stars which contains an excess of large parallaxes will also be negative, while for a group which contains an excess of small parallaxes the correction may be negative, zero, or positive. Usually it will be negative, but in very special cases, for groups in which only small parallaxes occur, the correction may be positive. These conclusions are contradictory to those of Young and Harper, who find that the correction is always positive, irrespective of the distribution of stars in the list between the different parallaxes, and that it is larger for a group of stars with small parallaxes.

2. *A Numerical Example.*

Since the list of stars used by Young and Harper is not yet available, we shall apply formula (3) to the list of stars used by Rimmer and, as it is necessary to neglect the small and negative parallaxes, we

46

shall assume that stars whose trigonometrical parallaxes are less than 0″·005 are not used. The distribution of stars as regards parallax is as follows :—

Parallax	″·005 – ″·015	″·015 – ″·025	″·025 – ″·035	″·035 – ″·045	″·045 – ″·055	″·055 – ″·065
Number of stars	39	50	40	26	27	15

Parallax	″·065 – ″·075	″·075 – ″·125	> ″·125
Number of stars	14	32	14

Suppose that the absolute magnitude of each star is computed from its trigonometrical parallax and we pick out those stars which have a given spectral type, say G6 to G7, and an absolute magnitude between M and M + dM (say, to fix ideas, between − 1·0 and + 1·0), the group being such that in it the parallaxes are distributed in approximately the same way as in the complete list. We shall compute the correction for such a group. Let the number of stars in the group be n. Then, since the total number of stars is 257, we can put $f(\pi_1, S, M)dM = n/257$. The difference between $u(\pi)$ and $y(\pi)$ is small and will be neglected and h will be taken to be 48, corresponding to a probable error of 0″·01 in the trigonometrical parallaxes. We notice that the denominator in the expression (3) is equal to the number of stars in the group and is therefore n. The integral in the numerator can be evaluated by some method of numerical integration. Employing the method which corresponds to the Trapezoidal rule for evaluating single integrals,* we find that the numerator is equal to − ·07 n. Hence the correction to be added to the computed mean absolute magnitude is − ·07. This may be compared with the correction as found by Young and Harper. The average parallax of the group will be about the same as the average parallax of all the stars. Omitting the few stars which have parallaxes greater than 0″·125 and which contribute practically nothing to the correction of − ·07 magnitude found above, the average parallax of all the stars is found to be 0″·04, and the correction according to Young and Harper † will therefore be about + 0·2 magnitude.

To illustrate the effect of the variation of f, consider the group of 9 stars of type G6 and G7 with computed absolute magnitudes ·9, ·7, ·6, ·6, ·5, ·4, ·3, − ·6, − 1·0. These stars have parallaxes all of which are very nearly 0″·02. Here $2f(\pi_1, S, M) = 9/257$ in the neighbourhood of $\pi_1 = {}''{\cdot}02$ and is equal to zero at all other places. Performing the computation afresh, we find that the correction is a little less than + 0·01 magnitude.

In case f is constant, we can find an approximate value of the correction by using the expression $k(\pi - \pi_1)/\pi$ instead of $\log \pi - \log \pi_1$. The correction computed by this method will not be very far out, since the small values of π are left out in computing absolute magnitudes, and the large values of the error $\pi - \pi_1$ are multiplied by a very small factor

* A. C. Aitkin and G. L. Frewin, "The Numerical Evaluation of Double Integrals," *Proc. Edinburgh Mathematical Soc.*, **42**, 4, 1924.

† *Loc. cit.*, p. 38.

$e^{-h^2(\pi-\pi_1)^2}$. When this is done we can perform the integration with respect to π_1 analytically and the integration with respect to π can then be performed numerically. The correction by this method for the above group of n stars comes out to be $-0{\cdot}12$ magnitude.

3. *Mean Absolute Magnitude from the Mean Parallax.*

A second method of computing the mean absolute magnitude used by Young and Harper consists in taking

$$5+\overline{m}+5\log\overline{\pi}_0$$

as the mean absolute magnitude, the bars in the above expression indicating mean values. They state the correction to the computed absolute magnitude to be

$$5\,\overline{\log\pi_0}-5\log\overline{\pi}_0.$$

The correction, obviously, should be

$$5\,\overline{\log\pi}-5\log\overline{\pi}_0. \qquad (4)$$

Of the errors in the computed mean absolute magnitudes, we must carefully distinguish between those which are due to the accidental errors in the trigonometrical parallaxes and those due to an inadequate method of computing the mean. The former will vanish if all the π_0's be replaced by π's, the latter will not vanish by this change. The correction calculated in Art. 1 to the mean of the separately computed absolute magnitudes will vanish if the accidental errors vanish. The correction to the mean absolute magnitude computed from the mean of the parallaxes does not vanish if the accidental errors all become zero. The expression (4) becomes in this case

$$5\,\overline{\log\pi}-5\log\overline{\pi}.$$

This will be large if the range in the parallaxes π is large. This is illustrated by the example considered by the authors. The correction found,* on the assumption that the use of spectroscopic parallaxes in place of the true parallaxes for the computation of this correction is justifiable, is $-0{\cdot}60$ magnitude for a group of 13 stars with a range in the spectroscopic parallaxes from $0''{\cdot}009$ to $0''{\cdot}100$. The correction is reduced to $-0{\cdot}14$ magnitude if one star is omitted, reducing the range in the parallaxes to $0''{\cdot}009$ to $0''{\cdot}029$. The correction will be very large if one of the parallaxes π is very small. A star with a negative measured parallax has presumably a very small true parallax. Hence we conclude that this method of calculating the mean absolute magnitude from the mean trigonometrical parallax is not good for the following reasons:—

1. Against the advantage of being able to include negative parallaxes must be considered the disadvantage of a large and uncertain error introduced by the method of computation itself. (The

* *Loc. cit.*, p. 40.

correction (4) cannot be calculated, for the true parallaxes π are unknown.)

2. The inclusion of only those stars which have a reasonably small dispersion in M and m and hence in π, as has been recommended by the authors, has the disadvantage of further curtailing the already small number of stars in a group, and also of bringing together in certain groups an excess of stars with only positive or only negative accidental errors in the trigonometrical parallaxes.

The statement that the true mean absolute magnitude of a group lies between the values found by the first and second methods is now seen to be invalid, and therefore there seems to be no reason for using the second method.

None of these objections apply to the formula used by Edwards in his recent paper * on the spectroscopic parallaxes of B type stars :—

$$\overline{\mathrm{M}} = 5 + 5 \log \{n/\Sigma 10^{-0 \cdot 2m}\} + 5 \log \bar{\pi}_0.$$

Here n is the number of stars in the group, which is formed according to the intensities of certain lines in the spectra. Each star has thus approximately the same absolute magnitude, but no restriction is placed on the dispersion in m or π.

In conclusion, I wish to express my gratitude to Professor Sampson for his kind interest in the preparation of this note.

* *Monthly Notices*, 84, No. 5, 367, 1924 March.

PRINTED IN GREAT BRITAIN BY NEILL AND CO., LTD., EDINBURGH.

On the Numerical Solution of Integral Equations.

On the Numerical Solution of Integral Equations.

By Gorakh Prasad.

(*Read 1st February 1924. Received 1st February 1924*).

1. *Introductory.*

The numerical solution of Integral equations with variable upper limits has been investigated by Professor Whittaker.* In this investigation the nucleus, supposed to be given numerically by a table of single entry, is replaced by an approximate expression consisting of a finite number of terms, each term involving an exponential or simply a power of the variable, and then the solution is found as an analytical expression from which its numerical values may be computed. The numerical solution of integral equations with fixed limits has been discussed by H. Bateman.† Methods for solving differential equations numerically have long been known ‡ and extensive use of such methods has been made, specially for the calculation of "special perturbations" in Astronomy. The differential equations giving the forms of drops of fluid under the influence of capillary action have also been numerically solved by Bashforth and Adams.§ Methods for the numerical solution of differential equations from a somewhat different point of view have been investigated by Runge,‖ Heun,** Kutta,†† and Piaggio.‡‡ The aim of the present paper is to find a method for the numerical solution of integral equations on the lines of the methods for solving differential equations.

---

* *Proc. Royal Soc.*, XCIV (A), 1918, pp. 367–383.

† *Proc. Royal Soc.*, C (A), 1921, pp. 441–449.

‡ See, for example, Bond, *Proc. American Academy*, IV., 1849, pp. 189–203, or Encke, *Astronomische Jahrbuch für* 1858.

§ Bashforth and Adams: *An attempt to test the Theories of Capillary Action*, Cambridge, 1883.

‖ *Mathematische Annalen*, XLVI., 1895, pp. 167–178.

** *Zeitschrift für Math. u. Phys.*, XLV., 1900, pp. 23–38.

†† *Zeitschrift für Math. u. Phys.*, XLVI., 1901, pp. 435–453.

‡‡ *Phil. Mag.*, XXXVI. (6th Ser.), 1919, pp. 596–600.

The advantages of the method given in Art. 2 below seem to be the following:—

(1). It is not necessary to replace either the nucleus or the function outside the integral by an approximate expression.

(2). Compared with other methods, this method is much less laborious.

(3). The computed values are accurate practically up to the last figure retained, although, if extreme accuracy is desired, it will be safer to retain one more figure.

(4). The method is applicable whether or not the nucleus $K(x, \xi)$ in equation (1) below is a function of $x - \xi$.

(5). The method is applicable to integral equations whose analytical solution cannot be found by the usual methods, for example to non-linear integral equations.

I wish to express my gratitude to Professor Whittaker for his kind help and encouragement.

2. *The integral equation of the second kind with a variable upper limit.*

Consider the integral equation

$$\phi(x) = f(x) + \int_a^x K(x, \xi)\, \phi(\xi)\, d\xi, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $f(x)$ is a continuous function of x in the range $a \leqq x \leqq b$, given either numerically or by an analytical expression and $K(x, \xi)$ is a real function of x and ξ, continuous in both the variables in the range $a \leqq \xi \leqq x \leqq b$, given either numerically or by an analytical expression, and $\phi(x)$ is the unknown function whose values are to be determined in the range $a \leqq x \leqq b$. We suppose further that the first few differential coefficients of $f(x)$ and also those of $K(x, \xi)$ with respect to x and ξ are continuous in the above ranges. We proceed to find accurately the values of $\phi(x)$ for $x = a, a + w, a + 2w, \ldots$, where w is taken to be so small that the intermediate values of $\phi(x)$ may be interpolated with certainty. It should also be so small that differences of

$$K(x, a+rw)\,\phi\,(a+rw)$$

above a certain order, say, to fix ideas, the fourth, are negligible.

Now suppose that we have already calculated ϕ_0, ϕ_1, ϕ_2, ... ϕ_{n-1}, where ϕ_r denotes $\phi\,(a+rw)$, and we are next to calculate ϕ_n. Let the successive differences of these quantities be taken according to the following scheme:—

x	ϕ	$\Delta\phi$	$\Delta^2\phi$	$\Delta^3\phi$	$\Delta^4\phi$
...	...				
$a+(n-5)w$	ϕ_{n-5}				
		$\Delta\phi_{n-5}$			
$a+(n-4)w$	ϕ_{n-4}		$\Delta^2\phi_{n-5}$		
		$\Delta\phi_{n-4}$		$\Delta^3\phi_{n-5}$	
$a+(n-3)w$	ϕ_{n-3}		$\Delta^2\phi_{n-4}$		$\Delta^4\phi_{n-5}$
		$\Delta\phi_{n-3}$		$\Delta^3\phi_{n-4}$	
$a+(n-2)w$	ϕ_{n-2}		$\Delta^2\phi_{n-3}$		$(\Delta^4\phi_{n-4})$
		$\Delta\phi_{n-2}$			
$a+(n-1)w$	ϕ_{n-1}				

The general run of the differences $\Delta^4\phi$ will suggest a close guess to the value of $\Delta^4\phi_{n-4}$, say $(\Delta^4\phi_{n-4})$, which will lead to a provisional value of ϕ_n, say (ϕ_n). Let the true value of ϕ_n be $(\phi_n)+\eta$.

We must now evaluate the integral in (1) numerically. A suitable formula for this is

$$\begin{aligned}\frac{1}{w}\int_a^{a+rw} f(x)\,dx = {} & f_0+f_1+f_2+\ldots+f_r-\tfrac{1}{2}\,(f_r+f_0)\\ & -\tfrac{1}{12}(\Delta f_{r-1}-\Delta f_0)-\tfrac{1}{24}(\Delta^2 f_{r-2}+\Delta^2 f_0)\\ & -\tfrac{19}{720}(\Delta^3 f_{r-3}-\Delta^3 f_0)-\tfrac{3}{160}(\Delta^4 f_{r-4}+\Delta^4 f_0)\\ & -\tfrac{863}{60480}(\Delta^5 f_{r-5}-\Delta^5 f_0)-\tfrac{275}{24192}(\Delta^6 f_{r-6}+\Delta^6 f_0)\\ & -\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)\end{aligned}$$

Neglecting the differences of order higher than the fourth, let the values of

$$\int_a^{a+nw} K(a+nw,\, \xi)\, \phi(\xi)\, d\xi,$$

computed by formula (2) with the value (ϕ_n) of ϕ_n, be (I_n). Find $f_n + (I_n) - (\phi_n)$ and denote it by ϵ.

The true value of the integral is

$$(I_n) + wK_{n,n}\,\eta\,\{1 - \tfrac{1}{2} - \tfrac{1}{12} - \tfrac{1}{24} - \tfrac{19}{720} - \tfrac{3}{160}\},$$

or

$$(I_n) + w\,K_{n,n}\,\eta\,\tfrac{95}{288},\text{*}$$

where $K_{n,n}$ stands for $K(a+nw,\, a+nw)$.

Substituting this in the integral equation (1), we find

$$(\phi_n) + \eta = f_n + (I_n) + \tfrac{95}{288}\, wK_{n,n}\,\eta, \qquad \text{.... ....(3)}$$

or

$$\eta = \frac{\epsilon}{1 - \frac{95}{288}\, wK_{n,n}}. \qquad \text{...... ....(4)}$$

Hence we have the following theorem :—

Theorem I. *If the values of the solution of the integral equation*

$$\phi(x) = f(x) + \int_a^x K(x,\, \xi)\, \phi(\xi)\, d\xi,$$

where $K(x, \xi)$ *and* $f(x)$ *may be given numerically, have been computed for* $x = a,\ a+w,\ a+2w,\ \ldots\ a+(n-1)\,w$, *and are* $\phi_0,\ \phi_1,\ \phi_2,\ \ldots\ \phi_{n-1}$, *then its value for* $x = a+nw$, *viz.*, ϕ_n, *is given by*

$$\phi_n = f_n + (I_n) + cwK_{n,n}\frac{f_n + (I_n) - (\phi_n)}{1 - cwK_{n,n}},$$

where f_n *denotes* $f(a+nw)$,

$K_{r,s}$ *denotes* $K(a+rw,\, a+sw)$,

(ϕ_n) *is any assumed value of* ϕ_n,

$$(I_n) = w\,[u_0 + u_1 + u_2 + \ldots + u_{n-1} + (u_n) - \tfrac{1}{2}\,\{(u_n) + u_0\} - \tfrac{1}{12}\,\{(\Delta u_{n-1}) - \Delta u_0\} - \tfrac{1}{24}\{(\Delta^2 u_{n-2}) + \Delta^2 u_0\} - \ldots \textit{ up to the term involving the } r^{\text{th}} \textit{ differences}],$$

$$u_p = K_{n,p}\,\phi_p,\ (u_n) = K_{n,n}(\phi_n),\ (\Delta u_{n-1}) = (u_n) - u_{n-1},\ \ldots$$

---

* It may be useful to note that if only the differences up to the 2nd, 3rd, 4th, ... order are included in the evaluation of (I_n), this coefficient becomes $\frac{3}{8}$, $\frac{251}{720}$, $\frac{95}{288}$, $\frac{19087}{60480}$, $\frac{5257}{17280}$, etc., respectively. See Bashforth and Adams, *loc. cit.*, p. 20. In this work are given tables which greatly facilitate the calculation of

$$\tfrac{19}{720}\,(\Delta^3 f_{r-3} - \Delta^3 f_0),\ \tfrac{3}{160}\,(\Delta^4 f_{r-4} + \Delta^4 f_0),\ \ldots\ \tfrac{8183}{1036800}\,(\Delta^8 f_{r-8} + \Delta^8 f_0).$$

and c is a numerical constant whose value is ·3299, ·3156 or ·3042, *according as the order r of the highest order difference used in the evaluation of* (I_n) *is* 4, 5 *or* 6.

It is easy to see that η need not be a small quantity in order that ϕ_n may be found correctly. In fact we may put $(\phi_n)=0$ and then find η, that is, ϕ_n, from (4). The procedure outlined above, however, saves a great deal of unnecessary labour, for with (ϕ_n) put equal to zero, the differences will all be large numbers, the 4th difference in this case being $-(\phi_{n-1}+\Delta\phi_{n-2}+\Delta^2\phi_{n-3}+\Delta^3\phi_{n-4})$. There is another point which should be mentioned here. We need not, if we prefer it, form a table of differences of ϕ to find an approximate value of ϕ_n. For, for the numerical evaluation of the integral $\int_a^{a+nw} K(a+nw, \xi)\,\phi(\xi)\,d\xi$, we shall have to form a table of the differences of the integrand, which will enable us to find an approximate value of $K_{n,n}\,\phi_n$. The table of differences of ϕ, however, serves as a useful check against accidental errors being made in the work, and, moreover, such a table is useful for interpolating intermediate values of ϕ.

Equation (3) shows that the true value of (ϕ_n), viz. $(\phi_n)+\eta$, differs from $f_n+(I_n)$ merely by $\frac{95}{288}\,wK_{n,n}\,\eta$, *i.e.* an error in the assumed value of ϕ_n gives rise to a much smaller error in the value of ϕ_n calculated from the integral equation (1) by using this assumed value for the evaluation of the integral occurring in it. In some cases $\frac{95}{288}\,w\,K_{n,n}\,\eta$ may be negligible and then we shall have simply

$$\phi_n=f_n+(I_n),$$

but in many cases at least $\frac{1}{288}\,wK_{n,n}\,\epsilon$ and $w^2\,K_{n,n}^2\,\epsilon$ will be negligible, and then we can write

$$\phi_n=f_n+(I_n)+\tfrac{1}{3}\,wK_{n,n}\,\epsilon. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

It is interesting to notice that η produces no error in the calculated value of ϕ_n if $K_{n,n}$ is zero. If on the other hand, $K_{n,n}$ is very large, we must take w sufficiently small to secure that the divisor $1-\frac{95}{288}\,wK_{n,n}$ shall not unduly magnify the error of the omitted decimals in (I_n).

3. *Initial values of the solution.*

It remains now to see how a few initial values of ϕ are to be calculated in order to start the solution. Obviously $\phi_0 = f_0$. To find the other values of ϕ, we choose a value of w, say w_1, where w_1 is so small that w_1^3 is negligible. Suppose that

$$\phi(a+w_1) = f(a+w_1) + \eta_1.$$

Then, applying the Trapezoidal rule for the evaluation of integrals to (1), we obtain a linear equation to find η_1, which gives

$$\eta_1 = \frac{\frac{1}{2}w_1\{K(a+w_1, a) f(a) + K(a+w_1, a+w_1) f(a+w_1)\}}{1 - \frac{1}{2}w_1 K(a+w_1, a+w_1)}. \quad \ldots\ldots\ldots(6)$$

Having thus calculated $\phi(a+w_1)$, we assume that

$$\phi(a+2w_1) = \phi(a+w_1) + \{\phi(a+w_1) - \phi(a)\} + \eta_2.$$

Applying this time Simpson's rule for evaluating the integral in (1), we again get a linear equation which gives η_2 and thus $\phi(a+2w_1)$ is found. If now $(2w_1)^5$ is negligible, we next calculate $\phi(a+4w_1)$ instead of $\phi(a+3w_1)$, Simpson's rule being again applied. If however $(2w_1)^5$ is not negligible, we must proceed more slowly; $\phi(a+3w_1)$ must also be calculated, and this time we may employ the Three-Eighths rule, or formula (2).* In this way we calculate $\phi(a+rw_1)$ for increasing values of r, always using the longest practicable interval between the successive ordinates to be summed and the best method of approximating to the integral as far as the materials in hand permit. In this way we shall soon have a sufficient number of known values of ϕ to employ the method given in Art. 2.

4. *An illustrative example.*

As an example of the method, let us solve the integral equation

$$\phi(x) = \frac{1}{1+x} - \frac{2}{2+x}\log_e(1+x) + \int_0^x \frac{1}{1+x-\xi}\phi(\xi)\,d\xi, \quad \ldots\ldots\ldots\ldots(7)$$

whose analytical solution can be seen to be $\phi(x) = \dfrac{1}{1+x}$.

---

* It must be borne in mind that the generalised Simpson's rule is less exact than formula (2) when we have calculated $\phi(a)$, $\phi(a+w)$, $\phi(a+2w)$ and $\phi(a+3w)$.

We notice that the successive derivatives of the nucleus rapidly increase, and the equation, therefore, is not one which will show the method of the present paper at its best.

Supposing that we wish to retain seven places of decimals, we choose w_1 to be ·005. Application of (6) gives us the correct value of $\phi(\cdot 005)$. By four successive applications of Simpson's rule we find $\phi(\cdot 01)$, $\phi(\cdot 02)$, $\phi(\cdot 04)$ and $\phi(\cdot 08)$. By applying the Three-Eighths rule we find $\phi(\cdot 12)$. With the help of (2), we now find $\phi(\cdot 16)$, $\phi(\cdot 20)$, $\phi(\cdot 24)$, ... $\phi(\cdot 40)$. Since we have now a sufficient number of values of ϕ to obtain differences up to the fifth order when the interval w is ·08, and since actual computation shows that the fourth and fifth differences are fairly small (omission of the fifth differences is found to affect the value of $\phi(\cdot 48)$ by less than half a unit in the eighth decimal place), we increase w to ·08 and thus calculate $\phi(\cdot 48)$ by using the values of $\phi(\xi)$ for $\xi = 0$, ·08, ·16, ·24, ·32 and ·40 only.

To illustrate the process of computation, suppose that values of $\phi(\xi)$ for $\xi = 0$, ·08, ·16, ... 2·00, have already been computed and that we want to compute $\phi(2\cdot 08)$. We form a table of the function $K(2\cdot 08, \xi)\,\phi(\xi)$ for $\xi = 0$, ·08, ·16, ... 2·00, and form the successive differences for six values of this function at the beginning and six or seven at the end. The latter part of the table is reproduced below.

ξ	$K(2\cdot 08, \xi)\phi(\xi)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
...	...					
1·52	·2543753					
		55000				
1·60	·2598753		11492			
		66492		1611		
1·68	·2665245		13103		469	
		79595		2080		163
1·76	·2744840		15183		632	
		94778		2712		217
1·84	·2839618		17895		849	
		112673		3561		(300)
1·92	·2952291		21456		(1149)	
		134129		(4710)		
2·00	·3086420		(26166)			
		(160295)				
2·08	(·3246715)					

From the fact that Δ^5 is increasing we put down 300 as an approximate value of the next fifth difference. This gives 1149 as the value of the next Δ^4 and so on, leading to $\cdot 3246715$ as an approximate value of $K(2\cdot 08, 2\cdot 08)\phi(2\cdot 08)$. Since $K(2\cdot 08, 2\cdot 08) = 1$, this is also the approximate value of $\phi(2\cdot 08)$. By an application of the formula (2) we now find (I) to be $\cdot 5514360$, and since $f(2\cdot 08) = -\cdot 2267608$, we find ϵ to be 37. Then (5) gives the value of $\phi(2\cdot 08)$ as $\cdot 3246753$, which is correct to the last place.

5. *Comparison with other methods.*

The heaviest part of the work consists in the various multiplications in order to tabulate $K(x, \xi)\,\phi(\xi)$, x remaining fixed for one integration, but varying from one integration to another. The work can be much shortened by using a larger value of the interval w. Thus in the above example we could have used $w = \cdot 16$ instead of $\cdot 08$ as we did, only in this case it would have been necessary to retain differences up to the ninth order. However, with a machine like the "Millionaire" for performing the multiplications, and an adding and listing machine (like a typewriter with the adding mechanism attached or a Burroughs adding and listing machine) to print the results of the multiplications and to add them automatically, combined with Bashforth's table giving the values of $\frac{19}{720}(\Delta^3 f_{r-3} - \Delta^3 f_0)$, ..., computations are quickly made.

The method given above appears to be a tedious and slow one, but this is partly due to the fact that we are working with seven decimal figures. If we desire, say, only two-place accuracy, we can find values of $\phi(\xi)$ from $\xi = 0$ to, say, $\xi = 6$ very quickly. Thus with $w = \cdot 25$ and an application of the Trapezoidal rule, we find $\phi(\cdot 25)$, then two successive applications of Simpson's rule give $\phi(\cdot 5)$ and $\phi(1\cdot 0)$ and ten more steps give the remaining values of ϕ, at intervals of half a unit, all correct to 2 decimal places.

This compares very favourably indeed with all the other methods. For, in order to use a method which requires an approximate representation of the nucleus by a polynomial, the first thing to do is to find this polynomial, but to represent the nucleus $\dfrac{1}{1+x}$ for $0 \leqq x \leqq 6$ correctly even only to 2 decimals, we shall have to use a polynomial of something like the sixth degree, a cubic being seen to give an error of 8 units in the second decimal place for some values of x. This will necessitate the solution of an algebraic equation of the sixth degree with coefficients involving two or three figures each and finally either a dozen numerical integrations or the representation of $\frac{1}{1+x}-\frac{2}{2+x}\log_e(1+x)$, supposed to be given numerically, by some approximate expression and the tabulation of the function, consisting of at least six terms, obtained as the result of the integration. Similarly, in view of the work required to represent a function, given numerically, by exponentials, a method using such a representation will be equally laborious. The solution in terms of iterated functions will be still more troublesome to compute. For seven-place accuracy these methods will naturally be far more tedious.

6. *Solution by a power-series.*

The method of solving integral equations indicated by the following theorem will sometimes be found useful.

Theorem II. *The solution of the integral equation*

$$\phi(x)=f(x)+\int_0^x K(x-\xi)\,\phi(\xi)\,d\xi, \qquad \text{..................(8)}$$

where the nucleus $K(x)$ and the function $f(x)$ are both supposed to be expansible in the Taylor's series *

$$K(x)=K_0+K_1x+K_2\frac{x^2}{2!}+K_3\frac{x^3}{3!}+\dots$$

and
$$f(x)=f_0+f_1x+f_2\frac{x^2}{2!}+f_3\frac{x^3}{3!}+\dots, \qquad \text{............(9)}$$

is given by

$$\phi(x)=\phi_0+\phi_1x+\phi_2\frac{x^2}{2!}+\phi_3\frac{x^3}{3!}+\dots, \qquad \text{..............(10)}$$

---

* Of course $f_1, f_2, \dots, \phi_1, \phi_2, \dots$ do not now have the same meaning as in Art. 2.

where ϕ_0, ϕ_1. ϕ_2, ... *are to be calculated successively from the equations*

$$\left.\begin{aligned}
&\phi_0 = f_0,\\
&\phi_1 = f_1 + K_0\,\phi_0,\\
&\phi_2 = f_2 + K_0\,\phi_1 + K_1\,\phi_0,\\
&\quad\dots\qquad\dots\qquad\dots\\
&\phi_r = f_r + K_0\,\phi_{r-1} + K_1\phi_{r-2} + \dots + K_{r-1}\,\phi_0.
\end{aligned}\right\}\quad\text{...........(11)}$$

Equations (11) are obtained by differentiating (8) repeatedly and then putting $x=0$.

It is easy to see that, under the supposition made above that $f(x)$ is expansible in a Taylor's series, this result is equivalent to the following theorem given by Professor Whittaker *:—

The solution of the integral equation

$$\phi(x) + \int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x),$$

where the nucleus $\kappa(x)$ is supposed to be expansible in a Taylor's series

$$\kappa(x) = \kappa_0 + \kappa_1 x + \kappa_2 \frac{x^2}{2\,!} + \kappa_3 \frac{x^3}{3\,!} + \dots,$$

is
$$\phi(x) = f(x) - \int_0^x K(x-s) f(s)\,ds, \quad\text{...............(12)}$$

where
$$K(x) = \kappa_0 - \begin{vmatrix} \kappa_0 & 1 \\ \kappa_1 & \kappa_0 \end{vmatrix} x + \begin{vmatrix} \kappa_0 & 1 & 0 \\ \kappa_1 & \kappa_0 & 1 \\ \kappa_2 & \kappa_1 & \kappa_0 \end{vmatrix} \frac{x^2}{2\,!}$$
$$- \begin{vmatrix} \kappa_0 & 1 & 0 & 0 \\ \kappa_1 & \kappa_0 & 1 & 0 \\ \kappa_2 & \kappa_1 & \kappa_0 & 1 \\ \kappa_3 & \kappa_2 & \kappa_1 & \kappa_0 \end{vmatrix} \frac{x_3}{3\,!} + \dots\text{(13)}$$

For, solving equations (11), we get

$$\phi_r = f_r + a_0 f_{r-1} + a_1 f_{r-2} + \dots + a_{r-1} f_0, \quad\text{...........(14)}$$

where, for brevity, a_r has been written for

$$\begin{vmatrix} K_0 & -1 & 0 & \dots & 0 \\ K_1 & K_0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ K_r & K_{r-1} & K_{r-2} & \dots & K_0 \end{vmatrix}$$

* *Loc. cit.*

Substituting the values of $K(x-s)$ and $f(s)$ from (13) and (9) in (12), multiplying out and integrating term by term, and remembering that

$$\int_0^x (x-\xi)^p \, \xi^q \, d\xi = \frac{p\,!\; q\,!}{(p+q+1)\,!}\, x^{p+q+1},$$

we find, after making the necessary changes of notation,

$$\phi(x) = f(x) + a_0 f_0 \, x + (a_0 f_1 + a_1 f_0) \frac{x^2}{2\,!} + \dots$$
$$+ (a_0 f_{r-1} + a_1 f_{r-2} + \dots + a_{r-1} f_0) \frac{x^r}{r\,!} + \dots,$$

which is the same as (10) by virtue of (14).

The solution in the form given in Theorem II. will generally be found more convenient, because no integration has to be performed and because it does not necessitate the evaluation of determinants.

The application of Theorem II. to equation (7) gives, when $0 \leqq x < 1$,

$$\phi(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$$

We can generalise Theorem II. as follows:—

Theorem III. *The solution of the integral equation*

$$\phi(x) = f(x) + \int_0^x K(x, \xi)\, \phi(\xi)\, d\xi,$$

where the nucleus $K(x, \xi)$ is supposed to be expansible in a Taylor's series

$$K(x, \xi) = K_{00} + (K_{10} x + K_{01} \xi)$$
$$+ \frac{1}{2\,!}(K_{20} x^2 + 2K_{11} x\xi + K_{02} \xi^2)$$
$$+ \frac{1}{3\,!}(K_{30} x^3 + 3K_{21} x^2 \xi + 3K_{12} x \xi^2 + K_{03} \xi^3) + \dots,$$

and $f(x)$ is supposed to be expansible in a Taylor's series,

$$f(x) = f_0 + f_1 x + f_2 \frac{x^2}{2\,!} + f_3 \frac{x^3}{3\,!} + \dots,$$

is given by

$$\phi(x) = \phi_0 + \phi_1 x + \phi_2 \frac{x^2}{2\,!} + \phi_3 \frac{x^3}{3\,!} + \dots,$$

where $\phi_0 = f_0$,

$$\phi_1 = f_1 + K_{00}\,\phi_0,$$

$$\phi_2 = f_2 + (2K_{10} + K_{01})\,\phi_0 + K_{00}\,\phi_1,$$

$$\phi_3 = f_3 + (3K_{20} + 3K_{11} + K_{02})\,\phi_0 + (3K_{10} + 2K_{01})\,\phi_1 + K_{00}\,\phi_2,$$

$$\cdots \quad \cdots \quad \cdots$$

and generally

$$\phi_r = f_r + \sum \frac{r\,!}{(r-p-q-1)\,!\,p\,!\,q\,!}\;\frac{1}{(p+q+1)!}\,K_{r-p-q-1,\,p}\,\phi_q,$$

in which the summation Σ *is to be extended over all positive integral values of* p *and* q, *including* 0, *such that*

$$r - p - q - 1 \geqq 0.$$

The proof of this theorem is similar to that of Theorem II.

7. *Non-linear integral equations and equations with an infinite nucleus.*

The method of Art. 2 is applicable, not only to the linear integral equation (1), but also to the more general integral equation

$$\phi(x) = F\left(x, \int_{f(x)}^{g(x)} K\{x, \xi, \phi(\xi)\}\,d\xi\right),$$

where $\phi(x)$ is the unknown function, whose values are to be determined for values of $x \geqq a$, and $F(x, y)$, $f(x)$, $g(x)$ and $K(x, y, z)$ are known functions, such that $a \leqq f(x) \leqq x$, $a \leqq g(x) \leqq x$, provided that certain conditions regarding the continuity of the functions F, K, f and g and certain of their differential coefficients are satisfied, and we are justified in assuming that a continuous solution exists. The procedure will be obvious from the following theorem regarding a simpler type of the non-linear integral equation, for which it is known that a solution exists under suitable conditions.*

* See Vergerio, *Annali di Matematica*, XXXI., 1922, pp. 81–119.

Theorem IV. *If the values of the solution of the integral equation*

$$\phi(x) = f(x) + \int_a^x K(x, \xi)\, F\{\phi(\xi)\}\, d\xi$$

have been computed for $x = a,\ a+w,\ \ldots\ a+(n-1)w$, *and are* $\phi_0, \phi_1, \phi_2, \ldots \phi_{n-1}$, *then its value for* $x = a + nw$, *viz.* ϕ_n, *is given by*

$$\phi_n = f_n + (I_n) + cwkK_{n,n}\ \frac{f_n + (I_n) - (\phi_n)}{1 - cwkK_{n,n}},$$

where the symbols have the same meaning as in Theorem I., except that now

> (ϕ_n) *is a value of* ϕ_n, *extrapolated from a table of differences of* ϕ, *so near to* ϕ_n *that squares and higher powers of* $\phi_n - (\phi_n)$ *may be neglected,*
>
> $u_p = K_{n,p}\ F\{\phi_p\}$, $(u_n) = K_{n,n}\ F\{(\phi_n)\}$,

and $k =$ *the value of* $\dfrac{dF(x)}{dx}$ *at* $x = (\phi_n)$.

The proof is similar to that of Theorem I. The initial values may be calculated almost exactly as before, the necessary changes being obvious, or they may be derived from the power series obtained by the method of Art. 6.

An interesting application can be made to the solution of the integral equation with an infinite nucleus of the type

$$\phi(x) = f(x) + \int_0^x \frac{G(x, \xi)}{(x-\xi)^{p/q}}\, \phi(\xi)\, d\xi \quad (0 < p < q), \quad \ldots\ldots\ldots(15)$$

where $G(x, \xi)$ satisfies the same conditions as $K(x, \xi)$ did in equation (1) and p and q are integers. Making a change of variables, (15) can be written as

$$\phi(x) = f(x) + q\int_0^{x^{1/q}} G(x, x - \xi^q)\phi(x - \xi^q)\, \xi^{q-p-1}\, d\xi,$$

which can be solved numerically like equation (1).

8. *An analogue of the formula of Kutta.*

The methods investigated by Runge and others for the solution of differential equations give corresponding methods for the solution of integral equations, but these methods are not so good as the method of Art. 2. A single example will suffice. Kutta's symmetrical formula, correct to the 4th order in w, gives the following result.

Theorem V. *If the values of the solution of the integral equation* (1) *have been computed for* $x=a,\ a+w,\ a+2w,\ \ldots\ a+(n-1)w$, *then its value for* $x=a+nw$ *may be computed from*

$$\phi_n = f_n + \sum_{r=0}^{n-1} \frac{\Delta'_r + 3\Delta''_r + 3\Delta'''_r + \Delta''''_r}{8}$$

or
$$\phi_n = f_n + \int_a^{a+(n-1)w} K(a+nw,\ \xi)\,\phi(\xi)\,d\xi + \frac{\Delta'_{n-1} + 3\Delta''_{n-1} + 3\Delta'''_{n-1} + \Delta''''_{n-1}}{8},$$

where
$$\Delta'_r = K_{n,\,r}\,\phi_r\,w,$$

$$\Delta''_r = K_{n,\,r+1/3}\left\{\phi_r + \frac{\Delta'_r}{3}\right\} w,$$

$$\Delta'''_r = K_{n,\,r+2/3}\left\{\phi_r + \Delta''_r - \frac{\Delta'_r}{3}\right\} w,$$

$$\Delta''''_r = K_{n,\,r+1}\left\{\phi_r + \Delta'''_r - \Delta''_r + \Delta'_r\right\} w,$$

and the other symbols have the same meaning as in Theorem I.

The second form of the formula is more convenient than the first, but it cannot be satisfactorily employed for small values of n.